MATH 3162 Homework Assignment 7 Solutions

7.2.3(a): $\Longrightarrow$: assume that f is integrable. Then, for every n, choose $\epsilon = \frac{1}{n}$, and by Theorem 7.2.8, there exists P_n so that $0 \le U(f,P_n) - L(f,P_n) < \frac{1}{n}$, which implies that $U(f,P_n) - L(f,P_n) \to 0$. In addition, since $\int_a^b f\ dx = U(f) = L(f)$, for every n we know that $L(f,P_n) \le \int_a^b f\ dx \le U(f,P_n)$. This means that for each n, $|\int_a^b f - L(f,P_n)| \le U(f,P_n) - L(f,P_n) < \frac{1}{n}$, so $L(f,P_n) \to \int_a^b f\ dx$; the proof that $U(f,P_n) \to \int_a^b f\ dx$ is similar.

$\Longleftarrow$: Assume that there exists a sequence of partitions P_n where $U(f,P_n) - L(f,P_n) \to 0$. Then for every ϵ, by definition of convergence, we can choose n so that $U(f,P_n) - L(f,P_n) = |U(f,P_n) - L(f,P_n)| < \epsilon$. Since ϵ was arbitrary, this proves that f is integrable. Also notice that $L(f,P_n) \le \int_a^b f\ dx \le U(f,P_n)$ for every n, and the proof that $L(f,P_n) \to \int_a^b f\ dx$ and $U(f,P_n) \to \int_a^b f\ dx$ is the same as in the paragraph above.

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7.2.3(b): For every n, define $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$, i.e. $x_k = \frac{k}{n}$. Then

$$L(f,P_n) = \sum_{k=0}^{n-1} m_k(x_{k+1}-x_k) = \sum_{k=0}^{n-1} \frac{k}{n}\frac{1}{n} = \frac{1}{n^2}(0+1+2+\ldots+(n-1)) = \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{n-1}{2n} \text{ and}$$

$$U(f,P_n) = \sum_{k=0}^{n-1} M_k(x_{k+1}-x_k) = \sum_{k=0}^{n-1} \frac{k+1}{n}\frac{1}{n} = \frac{1}{n^2}(1+2+\ldots+n) = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{n+1}{2n}.$$

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7.2.3(c): From (b), we see that $U(f,P_n) - L(f,P_n) = \frac{n+1}{2n} - \frac{n-1}{2n} = \frac{1}{n}$, which approaches 0. Therefore, $f(x) = x$ is integrable on $[0,1]$. Its integral $\int_0^1 x\ dx$ is the limit of $L(f,P_n)$, which is the limit of $\frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n}$, which is $\frac{1}{2}$.

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7.3.3: Define $f(x)$ as in the problem, equal to 1 iff $x = \frac{1}{n}$ for some n, and 0 otherwise. Choose any $\epsilon > 0$. By the Reverse Archimedean Principle, there exists N so that $\frac{1}{N} < \frac{\epsilon}{2}$. Start defining P by choosing $x_0 = 0$ and $x_1 = \frac{1}{N}$. Then, choose M so that $\frac{1}{M} < \frac{\epsilon}{2N}$, and define the rest of P by breaking up $[\frac{1}{N}, 1]$ into M equal pieces, i.e.

$$x_2 = \frac{1}{N} + \frac{1}{M}(1-1/N), x_3 = \frac{1}{N} + \frac{2}{M}(1-1/N), \ldots, x_{M+1} = \frac{1}{N} + \frac{M}{M}(1-1/N) = 1.$$

We now write

$$U(f,P) = \sum_{k=0}^{M} M_k(x_{k+1} - x_k) = M_0 \cdot \frac{1}{N} + \sum_{k=1}^{M} M_k \cdot \frac{1-1/N}{M} \le \frac{1}{N} + \sum_{k=1}^{M} M_k \frac{1}{M}.$$

Notice that there are only N numbers of the form $\frac{1}{n}$ which are greater than or equal to $x_1 = \frac{1}{N}$, namely $1, \frac{1}{2}, \ldots, \frac{1}{N}$. Therefore, there are at most N intervals $[x_k, x_{k+1}]$ for $k \geq 1$ where $M_k = 1$. This means that

$$U(f,P) \leq \frac{1}{N} + \sum_{k=1}^{M} M_k \frac{1}{M} \leq \frac{1}{N} + \frac{N}{M} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Finally, we note that

$$L(f,P) = \sum_{k=0}^{M} m_k(x_{k+1} - x_k) = 0,$$

since $m_k = 0$ for every k. This means that $U(f,P) - L(f,P) < \epsilon$, and since $\epsilon > 0$ was arbitrary, f is integrable. Finally, since $L(f,P) = 0$ for every P, $L(f) = 0$, and so $\int_0^1 f \, dx = 0$.

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7.3.9(a): Choose any $\epsilon > 0$. Since f is bounded, there exists M so that $-M \leq f(x) \leq M$ for all $x \in [a,b]$. Since D (the set of discontinuities of f) is content zero, there exist finitely many open intervals $I_1 = (a_1, b_1), \ldots, I_n = (a_n, b_n)$ so that $D \subseteq \bigcup I_k$ and the sum of the lengths of the I_k is less than $\frac{\epsilon}{4M}$. Without loss of generality, we assume that the I_k are disjoint and arranged in increasing order, i.e. $a < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < b$. Then f is continuous on D^c, meaning that it is continuous on the intervals $[a, a_1], [b_1, a_2], \ldots, [b_{n-1}, a_n], [b_n, b]$.

Since f is continuous on each of these intervals, it is integrable on each of them, and so we can assign partitions P_0 of $[a, a_1]$, P_1 of $[b_1, a_2]$, …, P_n of $[b_n, b]$ so that $U(f, P_k) - L(f, P_k) < \frac{\epsilon}{2(n+1)}$ for each k. Define P to be the union of all of the P_k. We now analyze $U(f,P) - L(f,P)$ by breaking it into pieces:

$$U(f,P) - L(f,P) = \sum_{k=0}^{N-1} (M_k - m_k)(x_{k+1} - x_k) =$$
$$\sum_{i=0}^{n} (U(f,P_i) - L(f,P_i)) + \sum_{i=0}^{n} (M^f_{[a_i,b_i]} - m^f_{[a_i,b_i]})(b_i - a_i).$$

The first piece represents the upper sum minus the lower sum over the intervals $[a, a_1]$, P_1 of $[b_1, a_2]$, …, P_n of $[b_n, b]$, which just comes from upper sums minus lower sums on the partitions $P_0, \ldots, P_n$ that we already defined for those intervals. The second piece represents the upper sum minus the lower sum just on the intervals $[a_1, b_1], \ldots, [a_n, b_n]$, which we did not split at all in P. ($m^f_{[a_i,b_i]}$ represents the inf of f over the interval $[a_i, b_i]$, and $M^f_{[a_i,b_i]}$ represents the sup of f over the same interval.) Since $-M \leq f(x) \leq M$ for all x, every $M^f_{[a_i,b_i]} \leq M$

and every $m^f_{[a_i,b_i]} \geq -M$. Therefore,

$$U(f,P) - L(f,P) = \sum_{i=0}^{n}(U(f,P_i) - L(f,P_i)) + \sum_{i=1}^{n}(M^f_{[a_i,b_i]} - m^f_{[a_i,b_i]})(b_i - a_i)$$

$$< (n+1)\cdot\frac{\epsilon}{2(n+1)} + \sum_{i=1}^{n} 2M(b_i - a_i) = \frac{\epsilon}{2} + 2M\sum_{i=1}^{n}(b_i - a_i) < \frac{\epsilon}{2} + 2M\frac{\epsilon}{4M} = \epsilon.$$

(The last $<$ comes from the fact that the sum of the lengths of the intervals $I_1, \ldots, I_n$ is less than $\frac{\epsilon}{4M}$.) Since ϵ was arbitrary, we've proved that f is integrable on $[a,b]$.

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7.3.9(b): If $F = \{x_1, \ldots, x_n\}$ is a finite set and $\epsilon > 0$, take $I_k = (x_k - \frac{\epsilon}{4n}, x_k + \frac{\epsilon}{4n})$ for $k = 1, 2, \ldots, n$. Then, since $x_k \in I_k$ for each k, obviously $F \subset \bigcup_{k=1}^{n} I_k$. Each I_k has length $\frac{\epsilon}{2n}$, so the sum of the lengths of the n intervals is $\frac{\epsilon}{2} < \epsilon$.

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7.3.9(c): Recall that the Cantor set is defined as $C = \bigcap_{n=0}^{\infty} C_n$, where $C_0 = [0,1]$, $C_1 = [0,1/3] \cup [2/3,1]$, and each C_n is obtained by removing the open middle thirds of the intervals in C_{n-1}. (So, for instance, $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$.) Notice that for every n, $C \subset C_n$. This is almost enough to finish, except that the definition of content zero requires OPEN intervals, and the intervals which comprise each C_n are closed.

So, we slightly change the definition: for each n, define C'_n by doubling the length of each small interval in C_n but keeping the same center, and making each open. So, for instance, $C'_1 = (-1/6, 1/2) \cup (1/2, 7/6)$. (The first interval in C'_1 has center $1/6$ and length $2/3$, compared to the first interval of C_1 which has center $1/6$ and length $1/3$, and the second interval of C'_1 is similarly defined.) Then for all n, $C_n \subset C'_n$, so $C \subset C'_n$. Also, C'_n is a union of 2^n open intervals which each have length $\frac{2}{3^n}$, so the total length is $2 \cdot (2/3)^n$. Fix any $\epsilon > 0$. Since $(2/3)^n \to 0$, we can choose N so that $(2/3)^N < \epsilon/2 \Longrightarrow 2 \cdot (2/3)^N < \epsilon$. Then, C'_N is a union of finitely many open intervals with total length less than ϵ which contains C, so by definition, C has content zero.

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7.3.9(d): Define f as in the problem, where $f(x) = 1$ if $x \in C$ and $f(x) = 0$ if $x \in C^c$. Recall that C is a closed set, so C^c is open. Therefore, for any $x \in C^c$, there exists $\delta > 0$ so that $(x - \delta, x + \delta) \subseteq C^c$, and so $f = 0$ on the interval $(x - \delta, x + \delta)$. This clearly implies that f is continuous at x. Since $x \in C^c$ was arbitrary, f is continuous on all of C^c. So, the discontinuity set D for f is contained in C, and therefore has content 0 by the argument given in part (c).

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7.4.4: Suppose that f is integrable and positive on $[a,b]$. By the Lebesgue Criterion from Section 7.6, the discontinuity set D of f has measure 0. So, D is not all of $[a,b]$. (If $[a,b]$ had measure 0, then the Dirichlet function would be integrable on $[a,b]$, and we know that's false.) Therefore, f is continuous at some point $c \in [a,b]$. Since f is positive, $f(c) > 0$. By choosing $\epsilon = f(c)/2$, we get δ so that for all $x \in [a,b]$ with $|x-c| < \delta$, $|f(x) - f(c)| < \epsilon \implies f(x) \in (f(c)-\epsilon, f(c)+\epsilon) \implies f(x) > f(c) - \epsilon = f(c)/2$. In other words, we have an entire interval $I = [y,z]$ so that $f(x) > f(c)/2$ for all $x \in I$. Then, since $f \geq 0$ on $[a,y]$ and $[y,b]$, and $f \geq f(c)/2$ on $[y,z]$,

$$\int_a^b f\ dx = \int_a^y f\ dx + \int_y^z f\ dx + \int_z^b f\ dx \geq 0 + (z-y)(f(c)/2) + 0 > 0.$$

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Written problem 1: This is extremely similar to the last problem. Since $f \geq 0$ for all $x \in [a,b]$ and f is not the zero function, there exists $c \in [a,b]$ such that $f(c) > 0$. Now, since f is continuous at c, we can apply exactly the same argument as in the previous problem to get that $\int_a^b f\ dx > 0$.

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Written problem 2: Note that from the conditions of the problem, $0 \leq f_n(x) \leq f_1(a)$ for all n and $x \in [a,b]$. Since each f_n is nonnegative for all x, $\int_a^b f_n\ dx \geq 0$ for all n. Now, choose any $\epsilon > 0$, and define $\delta = \frac{\epsilon}{2f_1(a)}$. From the conditions of the problem, $f_n(a+\delta) \to 0$, and so we can choose N so that for $n > N$, $|f_n(a+\delta) - 0| = f_n(a+\delta) < \frac{\epsilon}{2(b-a)}$. Then, for $n > N$, $f_n(x) \leq f_1(a)$ for $x \in [a, a+\delta]$ and $f_n(x) \leq f_n(a+\delta) < \frac{\epsilon}{2(b-a)}$ for $x \in [a+\delta, b]$ (recall that f_n is a decreasing function!) So, for all $n > N$,

$$\left|\int_a^b f_n\ dx - 0\right| = \int_a^b f_n\ dx = \int_a^{a+\delta} f_n\ dx + \int_{a+\delta}^b f_n\ dx \leq \delta f_1(a) + (b-a-\delta)\frac{\epsilon}{2(b-a)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves that $\int_a^b f_n\ dx \to 0$.

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Written problem 3: $\Longrightarrow$: Assume that f is ϵ-discontinuous at x. Then, for every n, choose $\delta = 1/n$. By definition, there exist $y, z \in (x - 1/n, x + 1/n)$ so that $|f(y) - f(z)| \geq \epsilon$; denote these as y_n and z_n. Then clearly $|f(y_n) - f(z_n)| \geq \epsilon$ for all n by definition. Also, since $x - 1/n < y_n, z_n < x + 1/n$ for all n, and $x - 1/n$ and $x + 1/n$ approach x (as $n \to \infty$), by the Squeeze Theorem, y_n and z_n both approach x.

$\Longleftarrow$: Suppose that sequences (y_n), (z_n) exist which both approach x and where for all n, $|f(y_n) - f(z_n)| \geq \epsilon$. Then, take any $\delta > 0$. By the definition of convergence, there exists N_1 so that when $n > N_1$, $|y_n - x| < \delta$. Similarly, there exists N_2 so that when $n > N_2$, $|z_n - x| < \delta$. Then, we can choose $N = \max(N_1, N_2) + 1$, and we know that $|y_N - x| < \delta$ and $|z_N - x| < \delta$, meaning that $y_N, z_N \in (x-\delta, x+\delta)$. Also, $|f(y_N) - f(z_N)| \geq \epsilon$, and so y_N and z_N can be the y and z required in the definition of ϵ-discontinuity, completing the proof.

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Written problem 4: We will instead prove the contrapositive, i.e. that if $f \neq g$ and f, g are continuous on $[a,b]$, then there exists h continuous on $[a,b]$ so that $\int fh \neq \int gh$. Suppose that f, g are continuous and $f \neq g$; then there exists $c \in [a,b]$ so that $f(c) \neq g(c)$. Without loss of generality, we assume that $f(c) > g(c)$ (otherwise, we could switch f and g to make this true.) Choose $\epsilon = \frac{f(c)-g(c)}{2}$ and use continuity of f, g at c to get $\delta_1, \delta_2 > 0$ so that $|x - c| < \delta_1 \Longrightarrow |f(x) - f(c)| < \epsilon$ and $|x - c| < \delta_2 \Longrightarrow |g(x) - g(c)| < \epsilon$. Then, define $\delta = \min(\delta_1, \delta_2)$. If $|x - c| < \delta$, then $|x - c| < \delta_1$ and $|x - c| < \delta_2$, and so

$$|f(x)-f(c)| < \epsilon \Longrightarrow f(x) \in (f(c)-\epsilon, f(c)+\epsilon) \text{ and } |g(x)-g(c)| < \epsilon \Longrightarrow g(x) \in (g(c)-\epsilon, g(c)+\epsilon).$$

Define I to be the interval of $x \in [a,b]$ for which $|x-c| < \delta$. Then, for all $x \in I$, $f(x) > f(c) - \epsilon$ and $g(x) < g(c) + \epsilon$, so $f(x) - g(x) > f(c) - g(c) - 2\epsilon = 0$ (check the definition of ϵ to confirm this).

Now, define h to be a nonnegative continuous function which is 0 for all x outside the interval I, and satisfies $h(c) > 0$. (This is easy to design; just picture h to be 0 on most of $[a,b]$, with a small "bump" contained entirely within I). Then, since $h = 0$ outside I and $h \geq 0$ and $f > g$ on I, $(f - g)h \geq 0$ on $[a,b]$. Finally, note that at $x = c$, $(f(c) - g(c))h(c)$ is positive since $f(c) > g(c)$ and $h(c) > 0$. So, $(f-g)h$ is continuous and nonnegative on $[a,b]$ and is not the zero function. Therefore, by Written Problem 1, $\int_a^b (f-g)h > 0$. But we can rewrite $\int_a^b (f-g)h = \int_a^b fh - gh = \int_a^b fh - \int_a^b gh > 0$, meaning that $\int_a^b fh \neq \int_a^b gh$, completing the proof.

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Alternate proof! Choose any f, g continuous on $[a,b]$ where $\int_a^b fh = \int_a^b gh$ for all continuous h on $[a,b]$. Then in particular, we can choose $h = f - g$, and so $\int_a^b f(f-g) = \int_a^b g(f-g)$, meaning that

$$0 = \int_a^b f(f-g) - \int_a^b g(f-g) = \int_a^b f(f-g) - g(f-g) = \int_a^b (f-g)^2.$$

Notice that $(f-g)^2$ is continuous on $[a,b]$ and nonnegative. Therefore, by the contrapositive of Written Problem 1, $(f-g)^2$ is the zero function, meaning that $f = g$.

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Written problem 5: Assume that f and g are integrable on $[a, b]$. Define D_f, D_g, and D_{fg} to be the discontinuity sets of f, g, and fg respectively. By the Lebesgue criterion, D_f and D_g have measure 0. Also, recall that if f and g are both continuous at some c, then fg is also continuous at c. Taking the contrapositive yields: fg discontinuous at $c \Longrightarrow f$ discontinuous at c or g discontinuous at c. This means that $c \in D_{fg} \Longrightarrow c \in (D_f \cup D_g)$, i.e. $D_{fg} \subseteq D_f \cup D_g$.

Now, choose any $\epsilon > 0$. Since D_f and D_g are measure zero, there exist countable sets of open intervals I_1, I_2, … and J_1, J_2, … such that $D_f \subseteq \bigcup I_n$, $D_g \subseteq \bigcup J_n$, and the sums of the lengths of the I_n and the lengths of the J_n are each less than $\epsilon/2$. But then we can just take the countable collection of intervals $I_1, J_1, I_2, J_2, \ldots$; the union contains $D_f \cup D_g$ and so contains D_{fg}, and the sum of the lengths is less than $\epsilon/2 + \epsilon/2 = \epsilon$. We have then shown that D_{fg} has measure zero. Then, by the Lebesgue criterion, fg is integrable on $[a, b]$.

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