Section 1.2:

5. As discussed in class, this problem reduces to showing that
\[ \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = 0. \]

There are many ways to do this. Perhaps the best is a change of variable; take \( u = \frac{1}{x^2} \). Then the limit can be rewritten as
\[ \lim_{u \to \infty} \frac{e^{-u}}{u^{1/2}} = \lim_{u \to \infty} \frac{u^{1/2}}{e^u}. \]

Use L’Hospital’s rule:
\[ \lim_{u \to \infty} \frac{u^{1/2}}{e^u} = \lim_{u \to \infty} \frac{u^{-1/2}/2}{e^u} = 0. \]

Any solution is basically leveraging the fact that the exponential term should “beat” the polynomial term.

6. A simple proof by induction (which I won’t include here) yields that for any \( n \) and any \( x > 0 \),
\[ f^{(n)}(x) = \frac{q_n(x)e^{-1/x^2}}{r_n(x)} \]

for some polynomials \( q_n(x), r_n(x) \). (For instance, \( f'(x) = \frac{e^{-1/x^2}}{2x^2} \) for all \( x > 0 \).)

Then, we prove by induction that \( f^{(n)}(x) = 0 \) for all \( n > 0 \). Problem 5 was the base case \( n = 1 \). Suppose we know that \( f^{(n)}(0) = 0 \). Then,
\[ f^{(n+1)}(0) = \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{f^{(n)}(x)}{x}. \]

The case \( x < 0 \) is trivial since \( f = 0 \) there, so we focus on \( x > 0 \);
\[ \lim_{x \to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0^+} \frac{e^{-1/x^2}}{x p_n(x)} = \lim_{x \to 0^+} \frac{(q_n(x))/(x r_n(x))}{e^{1/x^2}}. \]

For simplicity, we write \( q_{n,0}(x) = q_n(x) \) and \( r_{n,0}(x) = x r_n(x) \). Denote by \( s_0 \) the largest power of \( x \) that divides \( q_{n,0}(x) \) (clearly \( s_0 = 0 \) and by \( t_0 \) the largest power of \( x \) that divides \( r_{n,0}(x) \). Now, use L’Hospital’s Rule:
\[
\lim_{x \to 0^+} \frac{q_{n,0}(x)/r_{n,0}(x)}{e^{1/x^2}} = \lim_{x \to 0^+} \left( \frac{(q_{n,0}'(x)r_{n,0}(x) - q_{n,0}(x)r_{n,0}'(x))/((r_{n,0}(x))^2)}{-2x^{-5}e^{1/x^2}} \right)
\]
\[ = \lim_{x \to 0^+} \frac{(x^3 q_{n,0}'(x)r_{n,0}(x) - x^3 q_{n,0}(x)r_{n,0}'(x))}{e^{1/x^2}}. \]
Write \( q_{n,1}(x) = x^3q'_{n,0}(x)r_{n,0}(x) - x^3q_{n,0}(x)r'_{n,0}(x) \) and \( r_{n,1}(x) = 2(r_{n,0}(x))^2 \); then the largest power of \( x \) that divides \( q_{n,1}(x) \) is \( s_1 = s_0 + t_0 + 2 \). Similarly, the largest power of \( x \) that divides \( r_{n,1}(x) \) is \( t_1 = 2t_1 \). Note that \( t_1 - s_1 = t_0 - s_0 - 2 \). If we continue in this way, we will eventually arrive at \( q_{n,k}(x) \) and \( r_{n,k}(x) \) where the largest power of \( x \) dividing \( q_{n,k} \) is larger. Then,

\[
 f^{n+1}(x) = \lim_{x \to 0^+} \frac{q_{n,0}(x)/r_{n,0}(x)}{e^{x/2}} \]

is the limit of an expression whose numerator approaches 0 and denominator approaches infinity, and so it is 0.

Section 1.3:

3(a). Clearly \( x = 0 \) is a fixed point, and for any \( x \neq 0 \), \( |f(x)| = |x| < |x| \). Therefore, for any \( x \neq 0 \), \( |f^n(x)| < |x| \) for all \( n > 0 \), so \( x \) is not periodic. Therefore, \( x = 0 \) is the only periodic point.

3(b). Clearly \( x = 0 \) is a fixed point, and for any \( x \neq 0 \), \( |f(x)| = 3|x| > |x| \). Therefore, for any \( x \neq 0 \), \( |f^n(x)| > |x| \) for all \( n > 0 \), so \( x \) is not periodic. Therefore, \( x = 0 \) is the only periodic point.

3(c). Clearly \( x = 0 \) is a fixed point, and for any \( x \in (0,1] \), \( f(x) = x - x^2 < x \), but \( f(x) = x(1-x) \geq 0 \). Therefore, for any \( x \neq 0 \), \( f^n(x) < x \) for all \( n > 0 \), so \( x \) is not periodic. Therefore, \( x = 0 \) is the only periodic point.

3(f). Clearly \( x = -1, 0, 1 \) are all fixed points. For any \( x \in (0,1) \), \( f(x) = \frac{1}{2}(x + x^3) < \frac{1}{2}(2x) = x \), and \( f(x) > 0 \). Therefore, for any \( x \in (0,1) \), \( f^n(x) < x \) for all \( n > 0 \), so \( x \) is not periodic. Similarly, for any \( x \in (-1,0) \), \( f(x) = \frac{1}{2}(x + x^3) > \frac{1}{2}(2x) = x \), and \( f(x) < 0 \). Therefore, for any \( x \in (-1,0) \), \( f^n(x) > x \) for all \( n > 0 \), so \( x \) is not periodic. Therefore, \( x = -1, 0, 1 \) are the only periodic points.

4(a). The stable set of 0 is \( \mathbb{R} \); for any \( x \neq 0 \), \( f^n(x) = (-1/2)^n x \), which approaches 0 as \( n \to \infty \).

4(b). The stable set of 0 is \( \{0\} \); for any \( x \neq 0 \), \( |f^n(x)| = 3^n x \), which approaches \( \infty \) as \( n \to \infty \), and therefore \( f^n(x) \) does not approach 0.

4(c). The stable set of 0 is \( [0,1] \); as shown before, \( f^n(x) \) is decreasing and nonnegative for all \( x \in [0,1] \), and so must approach a limit. But that limit must then be a fixed point, and 0 is the only one. Since \( f^n(x) \) approaches 0 for all \( x \in [0,1] \), \( [0,1] \) is the stable set of 0.

4(f). The stable set of 0 is \( (0,1) \), the stable set of \(-1\) is \( \{-1\} \), and the stable set of 1 is \( \{1\} \). To prove this, it suffices to show that for all \( x \in (-1,1) \), \( f^n(x) \)
approaches 0. Recall that if \( x \in (0, 1) \), then \( f^n(x) \) is decreasing and positive, and therefore it approaches a limit. This limit must be a fixed point, and 0 is the only possible value. Therefore, \( f^n(x) \to 0 \). A similar argument shows that for \( x \in (-1, 0) \), \( f^n(x) \to 0 \) as well. Finally, 0 is a fixed point, so trivially \( f^n(0) \to 0 \).

7. If \( f \) is a homeomorphism, then as discussed in class, either \( f \) is increasing on all of \( \mathbb{R} \) or decreasing on all of \( \mathbb{R} \). If \( f \) is increasing, then for all \( x < y \), \( f(x) < f(y) \). Suppose that \( x \) is not a fixed point of \( f \). Then, either \( f(x) > x \) or \( f(x) < x \). We treat only the first case, as the second is trivially similar. Since \( f(x) > x \) and \( f \) is increasing, \( f(f(x)) = f^2(x) > f(x) \). By induction, the orbit \( f^n(x) \) is increasing. But then clearly \( f^n(x) > x \) for all \( n \), so \( x \) is not periodic.

We have shown that the only possible periodic points for increasing \( f \) are fixed points.

Now, suppose that \( f \) is decreasing, i.e. for all \( x < y \), \( f(x) > f(y) \). Then clearly \( f^2 \) is increasing; if \( x < y \), then \( f(x) > f(y) \), implying that \( f^2(x) < f^2(y) \). Suppose that \( f \) has a periodic point \( x \), i.e. \( f^n(x) = x \). Then clearly \( (f^2)^n(x) = f^{2n}(x) = (f^n)^2(x) = x \), so \( x \) is a periodic point of \( f^2 \) as well. But since \( f^2 \) is increasing, this means that \( x \) is a fixed point of \( f^2 \), so \( f^2(x) = x \) and \( x \) has least period either 1 or 2. We’ve shown that the only possible periodic points for decreasing \( f \) have least period 1 or 2.

There are many examples we could use here; for instance, \( f(x) = -x \) is a homeomorphism (it’s its own inverse), but every nonzero \( x \) has least period 2 for \( f; f^2(x) = -(f(x)) = -(-x) = x \).

10. We claim that every number of the form \( x = \frac{i}{2^n} \) for positive integers \( i, n \) is periodic for \( f \). To see this, simply note that \( f^n(x) = 2^n x \) (mod 1) = \( \frac{2^n i}{2^n - 1} \) (mod 1) = \( \frac{2^n i - i + i}{2^n - 1} \) (mod 1) = \( \frac{i}{2^n - 1} \) (mod 1) = \( \frac{i}{2^n - 1} \).

The proof that the set of such numbers is dense is extremely similar to the one for \( \mathbb{Q} \) and we won’t include it here. The only relevant fact is that we have denominators approaching infinity, with all possible integer numerators.

11. We claim that every number of the form \( x = \frac{i}{2^n} \) for positive integers \( i, n \) is eventually fixed for \( f \). To see this, simply note that \( f^n(x) = 2^n x \) (mod 1) = \( \frac{2^n i}{2^n} \) (mod 1) = \( i \) (mod 1) = 0, and that 0 is fixed for \( f \).

The proof that the set of such numbers is dense is extremely similar to the one for \( \mathbb{Q} \) and we won’t include it here. The only relevant fact is that we have denominators approaching infinity, with all possible integer numerators.