MATH 3451 Homework Assignment 6 Solutions

1. We proved the following fact in class and called it Lemma 1: if $I \to J$, then there exists $I' \subseteq I$ with $f(I') = J$. Use that fact to prove the following, which we called Lemma 2: if $I_0 \to I_1 \to I_2 \cdots \to I_n$, then there exists $I_0' \subseteq I_0$ for which $f^i(I_0') \subseteq I_i$ for $0 \leq i \leq n$ and $f^n(I_0') = I_n$.

Solution: We proceed by induction on n. The case $n = 1$ is just Lemma 1, which was proved in class. Assume that the statement holds for some n, and we will prove it for $n + 1$. So, assume that we have a chain of intervals $I_0 \to I_1 \to I_2 \cdots \to I_n \to I_{n+1}$. By Lemma 1, there exists $I_n' \subseteq I_n$ so that $f(I_n') = I_{n+1}$. Since $I_n' \subseteq I_n$, the chain

$$I_0 \to I_1 \to \cdots I_{n-1} \to I_n'$$

is still valid. Therefore, by the inductive hypothesis, there exists $I_0' \subseteq I_0$ so that $f^i(I_0') \subseteq I_i$ for $0 \leq i \leq n$ and $f^n(I_0') = I_n'$. Furthermore, $f^{n+1}(I_0') = f(f^n(I_0')) = f(I_n') = I_{n+1}$. Therefore, $f^i(I_0') \subseteq I_i$ for $0 \leq i \leq n+1$ and $f^{n+1}(I_0') = I_{n+1}$, completing the proof for $n + 1$. This completes the induction, so we are done.

2. Give an example of a function f with points of every least period.

Solution: There are many examples. For instance, simply define a piecewise linear function with $f(1) = 2$, $f(2) = 3$, and $f(3) = 1$. Then clearly f has a point of least period 3 and is continuous, so f has points of every least period by Sharkovsky's Theorem.

3. Prove the following facts about the relationship between least periods of f^2 and least periods of f: (these will be very useful for problems 4 and 5!)

- If a point x has least period $2n$ for f, then x has least period n for f^2.
- If a point x has least period n for f^2 and n is EVEN, then x has least period $2n$ for f.
- If a point x has least period n for f^2 and n is ODD, then x has least period either $2n$ for f or least period n for f.

Proof: We begin by noting that x has least period $2n$ for f if and only if $f^{2n}(x) = x$ and $f(x), f^2(x), \ldots, f^{2n-1}(x) \neq x$. Similarly, x has least period n for f^2 if and only if $f^{2n}(x) = x$ and $f^2(x), f^4(x), \ldots, f^{2n-2}(x) \neq x$. From this, the first bullet point above is obvious, since the set $\{f(x), f^2(x), \ldots, f^{2n-1}(x)\}$ contains the set $\{f^2(x), f^4(x), \ldots, f^{2n-2}(x)\}$.

Now, assume n is even, and that x has least period n for f^2. Then, $f^i(x) \neq x$ for all even i with $1 < i < 2n$; in particular, $f^n(x) \neq x$ since n is even. To prove x has least period $2n$ for f, we need only to prove that $f^i(x) \neq x$ for all odd i less than $2n$. Consider any such i. There are two cases: either $1 \leq i < n$

or $n < i < 2n$. (i cannot equal n since n is even!) If $f^i(x) = x$, then clearly $f^{2i}(x) = f^i(f^i(x)) = x$. If $1 \leq i < n$, then $2i$ is even and less than $2n$, and so we have a contradiction. If $n < i < 2n$, then $2i > 2n$, so we don't have an immediate contradiction. However, we can write $f^{2i}(x) = f^{2i-2n}(f^{2n}x) = f^{2i-2n}(x)$, and $2i - 2n$ is even and less than $2n$, and so we again have a contradiction. We've then shown that $f^i(x) \neq x$ for all odd i less than $2n$, so x has least period $2n$ for f.

Finally, assume n is odd, and that x has least period n for f^2. Again, $f^i(x) \neq x$ for all even i with $1 < i < 2n$. The only change in the above proof is now that it is possible to have $i = n$ (since n is odd), and so the case $f^n(x) = x$ does not lead to a contradiction. This means that the least period for x under f in this case is either $2n$ OR n, which is exactly what we were trying to prove.

4. Prove Sharkovsky's Theorem for all k of the form $2^n j$, j odd and greater than 1, from the proof of Sharkovsky's Theorem for k odd and greater than 1 given in class.

Proof: We begin with the case $n = 1$. Consider any number of the form $k = 2j$, j odd and greater than 1, and assume that f has a point x of least period $k = 2j$. Then, by problem 1.5, x has least period j for f^2. Then, since j is odd, by the proof of Sharkovsky's Theorem for odd numbers, we know that f^2 has points of all least periods less than j in the Sharkovsky ordering, i.e. all integers in the set $S = \{1, 2, 4, 6, 8, \ldots, j-1, j, j+1, j+2, \ldots\} = \mathbb{N} \setminus \{3, 5, 7, \ldots, j-2\}$.

Then, we want to go back and conclude something about least periods for points under f. For any even element m of S, problem 1.5 implies that since f^2 has a point of least period m, f has a point of least period $2m$. But for odd elements m of S other than 1 (such as $j+2$), all that we know from problem 3 is that since f^2 has a point of least period m, f has either a point of least period m or a point of least period $2m$. However, since m is odd and not 1 and $2m$ is even, $2m$ is less than m in the Sharkovsky ordering. Therefore, if f has a point of least period m, it also has a point of least period $2m$ by the proof of Sharkovsky's Theorem for odd k. This means that no matter what, for every $m \in S$ except $m = 1$, f has a point of least period $2m$. Therefore, f has points of all least periods in the set $2(S \setminus \{1\}) = 2\mathbb{N} \setminus \{2, 6, 10, 14, \ldots, 2(j-2)\}$. Finally, we note that this is almost exactly the set of numbers less than $k = 2j$ in the Sharkovsky ordering; the only difference is that $1, 2$ are less than $2j$ and $1, 2 \notin 2(S \setminus \{1\})$. However, we proved in class that any function with a periodic nonfixed point has points of least periods $1, 2$, and so f has points of least periods $1, 2$ as well. Since $2(S \setminus \{1\}) \cup \{1, 2\} = 2S \cup \{1\}$ is exactly the set of integers less than $k = 2j$ in the Sharkovsky ordering, and we've proved that all numbers in $2S \cup \{1\}$ are least periods of some points under f, we've proved Sharkovsky's theorem for $k = 2j$.

Now, we need a proof for larger n. Assume that we've already proved Sharkovsky's theorem for all numbers of the form $k = 2^n j$, $n \geq 1$. Consider

$k = 2^{n+1}j$, j odd and greater than 1, and assume that f has a point x of least period $k = 2^{n+1}j$. Again, this means that x has least period $2^n j$ under f^2, and so that f^2 has points of all least periods less than $2^n j$ in the Sharkovsky ordering by the assumption that we've proved Sharkovsky's theorem for numbers of the form $k = 2^n j$. Denote by S the set of all integers less than $2^n j$ in the Sharkovsky ordering. Since $n \geq 1$, $2^n j$ is even, so all elements of S are even except for 1. Therefore, by problem 3, for any element m of $S \setminus \{1\}$, since f^2 has a point of least period m, f has a point of least period $2m$. We've then shown that f has points of all least periods in $2(S \setminus \{1\})$. Again, it's easily checked that the set of numbers less than $2^{n+1}j$ in the Sharkovsky ordering (i.e. the set of least periods we are trying to achieve) is just $2S \cup \{1\} = 2(S \setminus \{1\}) \cup \{1, 2\}$. We then need only verify that f has points of least periods 1 and 2, but we note again that this special case was proved in class. We've then shown that f has points of all least periods less than $k = 2^{n+1}j$ in the Sharkovsky ordering, completing the induction and the proof.

5. Prove Sharkovsky's Theorem for all k of the form 2^n by using the fact, proved in class, that any f with a non-fixed periodic point contains a point of least period 2.

Proof: We proceed by induction. For $n = 0$, Sharkovsky's theorem says nothing since $2^n = 1$ is the smallest number in the Sharkovsky ordering already. For $n = 1$, Sharkovsky's theorem states only that if f has a point of least period 2, then it has a fixed point, which is a corollary of a result proved in class. Now, suppose that $n \geq 1$ and that Sharkovsky's theorem holds for 2^n, i.e. that if a function f has a point of least period 2^n, then it automatically has points of all least periods 2^j for $j < n$. We wish to prove Sharkovsky's theorem for 2^{n+1}, so assume that f is a continuous function and that f has a point x of least period 2^{n+1}. Then, by problem 3, x has least period 2^n for f^2. Then, by the inductive hypothesis, there are points with all least periods 2^j for $j < n$ for f^2. Then, if $j > 0$, a point y with least period 2^j for f^2 is a point with least period 2^{j+1} for f, since 2^j is even. Therefore, f has points of all least periods $2 \cdot 2^j = 2^{j+1}$ for $0 < j < n$, or all least periods 2^k for $1 < k < n + 1$. It remains only to show that f has points with least periods $2^0 = 1$ and $2^1 = 2$. However, $n \geq 1$, so $2^n \geq 2$, so we assumed that f had a point of least period $2^n \geq 2$. From results in class, this implies that f has points of least periods 2 and 1, completing the induction, and the proof.