Homework 3 - Solutions

Complex Variables

Section 20, Problem 1. By definition, $f'(z_0)$ is

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{\left(z_0^2 + 2\, z_0\, \Delta z \;+\; \Delta z^2\right) - z_0^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2\, z_0\, \Delta z \;+\; \Delta z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2\, z_0 \;+\; \Delta z$$

$$= 2\, z_0.$$

○

Section 20, Problem 2(d). We have:

$$\frac{d}{dz}\left(\frac{\left(1+z^2\right)^4}{z^2}\right) = \frac{z^2 \frac{d}{dz}\left(1+z^2\right)^4 - \left(1+z^2\right)^4 \frac{d}{dz}\left(z^2\right)}{\left(z^2\right)^2} = \frac{4\, z^2\left(1+z^2\right)^3 (2\, z) - \left(1+z^2\right)^4 (2\, z)}{z^4}$$

$$= \frac{8\, z^3\left(1+z^2\right)^3 - 2\, z\left(1+z^2\right)^4}{z^4}.$$

○

Section 20, Problem 4. Since $f'(z_0)$ and $g'(z_0)$ exist, and $g'(z_0) \neq 0$, we know that

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}{\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}}.$$

But $f(z_0) = g(z_0) = 0$ implies

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)}{z - z_0} \qquad \text{and} \qquad \lim_{z \to z_0} \frac{g(z)}{z - z_0}.$$

Therefore,

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{z \to z_0} \frac{f(z)/(z - z_0)}{g(z)/(z - z_0)} = \lim_{z \to z_0} \frac{f(z)}{g(z)}.$$

○

Section 20, Problem 8(a). First,

$$\frac{\Delta f}{\Delta z} := \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\mathrm{Re}(z + \Delta z) - \mathrm{Re}(z)}{\Delta z} = \frac{\mathrm{Re}(z) + \mathrm{Re}(\Delta z) - \mathrm{Re}(z)}{\Delta z} = \frac{\mathrm{Re}(\Delta z)}{\Delta z},$$

where $\Delta z = \Delta x + i \Delta y$. If $f'(z) = \lim_{\Delta z \to 0} \Delta f / \Delta z$ exists, it must have the same value independently of the path in which we approach the origin.

If we approach the origin through the real axis, then $\mathrm{Re}(\Delta z) = \Delta x = \Delta z$, and consequently $\lim_{\Delta z \to 0} \Delta f / \Delta z = 1$. On the other hand, if we approach the origin through the imaginary axis, then $\operatorname{Re}(\Delta z) = 0$, and therefore $\lim_{\Delta z \to 0} \Delta f / \Delta z = 0$. Hence, $f'(z)$ does not exists at any point.

○

Section 20, Problem 9. Suppose that $z = 0$. For $\Delta z = \Delta x + i\Delta y$ we have

$$\frac{\Delta f}{\Delta z} := \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{f(0+\Delta z) - 0}{\Delta z} = \frac{f(\Delta z)}{\Delta z} = \frac{(\overline{\Delta z})^2}{(\Delta z)^2}.$$

If Δz lies on the real axis, then $\Delta z = \Delta x$, and so $\Delta f / \Delta z = (\overline{\Delta x})^2 \big/ (\Delta x)^2 = 1$; if Δz lies on the imaginary axis, then $\Delta z = i\Delta y$ and so $\Delta f / \Delta z = (\overline{i\Delta y})^2 \big/ (i\Delta y)^2 = 1$. Therefore, $\Delta f / \Delta z = 1$ over the real and imaginary axes. On the other hand, when Δz lies on the line $\Delta x = \Delta y$, then $\Delta z = \Delta x + i\Delta x$, and

$$\frac{\Delta f}{\Delta z} = \frac{(\overline{\Delta z})^2}{(\Delta z)^2} = \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2} = \frac{-2\,i(\Delta x)^2}{2\,i(\Delta x)^2} = -1.$$

Hence, $f'(0)$ does not exist since it is not independent of the path in which we approach the origin.

○

Section 24, Problem 1(d).
Let $z := x + iy \in \mathbb{C}$. Then, $f(z) := e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x \cos y - i\,e^x \sin y$. Therefore, $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$. Then, $u_x = e^x \cos y$ and $v_y = -e^x \cos y$. The only way for $u_x = v_y$ to hold is then if $\cos y = 0$. Similarly, $u_y = -e^x \sin y$ and $v_x = -e^x \sin y$, so the only way for $u_y = -v_x$ to hold is if $\sin y = 0$. It is not possible for both cos y = 0 and $\sin y = 0$ to be true simultaneously, and so the Cauchy-Riemann equations are not satisfied at any point. Therefore, $f'(z)$ does not exist at any point.

○

Section 24, Problem 3(b). Let $z_0 := x_0 + iy_0 \in \mathbb{C}$. Write $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = x^2$ and $v(x, y) = y^2$. Then, f is defined everywhere and $u_x = 2x$, $u_y = 0 = v_x$, and $v_y = 2y$.

By the theorem in Section 23, we know that $f'(z_0)$ exists, and equals $u_x(x_0, y_0) + iv_x(x_0, y_0)$, if the first-order partial derivatives of u and v:

(a) exist in some neighborhood of (x_0, y_0);
(b) are continuous at (x_0, y_0);
(c) and satisfy the Cauchy-Riemann equations at (x_0, y_0).

Clearly, (a) and (b) are satisfied. However, (c) is satisfied if and only if $u_x(x_0, y_0) = 2x_0 = 2y_0 = v_y(x_0, y_0)$, which in turn is equivalent to $x_0 = y_0$. Therefore, $f'(z)$ exists for every $z_0 \in \mathbb{C}$ where $z_0 = x_0 + ix_0$, and in such a case $f'(z_0) = u_x + iv_x = 2x_0$.

○

Section 24, Problem 6. Recall from page 69 that $u_r = u_x \cos\theta + u_y \sin\theta$, $u_\theta = -u_x r\sin\theta + u_y r\cos\theta$, $v_r = v_x \cos\theta + v_y \sin\theta$, and $v_\theta = -v_x r\sin\theta + v_y r\cos\theta$. So, $u_r + i\,v_r = u_x \cos\theta + u_y \sin\theta + i(v_x \cos\theta + v_y \sin\theta)$. Since the Cauchy-Riemann equations hold, we can rewrite this as $u_r + i\,v_r = u_x \cos\theta - v_x \sin\theta + i\,v_x \cos\theta + i\,u_x \sin\theta = (u_x + i\,v_x)(\cos\theta + i\sin\theta) = (u_x + i\,v_x)\,e^{i\theta}$.
Therefore, $u_x + i\,v_x = (u_r + i\,v_r)\,e^{-i\theta}$. We already know that f' can be written as $u_x + i\,v_x$, so we're done.