Solutions for MATH 3851 Homework Assignment 4

Textbook problems:

Section 24, 2(b): $f(z) = e^{-x}e^{-iy} = e^{-x}(\cos(-y) + i\sin(-y)) = e^{-x}\cos y - ie^{-x}\sin y$. So, $u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$. Then $u_x = -e^{-x}\cos y$, $u_y = -e^{-x}\sin y$, $v_x = e^{-x}\sin y$, and $v_y = -e^{-x}\cos y$.

Clearly, all four functions are continuous everywhere, and the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied everywhere in $\mathbb{C}$. So, f is entire, and $f' = u_x + iv_x = -e^{-x}\cos y + ie^{-x}\sin y$.

To find the second derivative, we break f' up; for f', $U = -e^{-x}\cos y$ and $V = e^{-x}\sin y$. (We use capital letters for f' to avoid confusion with the real and imaginary parts of the original function f.) Then $U_x = e^{-x}\cos y$, $U_y = e^{-x}\sin y$, $V_x = -e^{-x}\sin y$, and $V_y = e^{-x}\cos y$.

Again, all four functions are continuous everywhere, and the Cauchy-Riemann equations $U_x = V_y$ and $U_y = -V_x$ are satisfied everywhere in $\mathbb{C}$. So, f' is entire, and $f'' = (f')' = U_x + iV_x = e^{-x}\cos y - ie^{-x}\sin y$, which is in fact equal to the original function f.

Section 24, 3(c): We rewrite: $f(z) = z\text{Im}(z) = (\text{x} + \text{iy})(\text{y}) = \text{xy} + \text{iy}^2$. So, $u = xy$ and $v = y^2$. Then $u_x = y$, $u_y = x$, $v_x = 0$, and $v_y = 2y$.

These functions are all continuous everywhere in $\mathbb{C}$, but the Cauchy-Riemann equations are only satisfied if $u_x = v_y$ and $u_y = -v_x$, meaning that $y = 2y$ and $x = 0$, or $x = 0$ and $y = 0$. So, f is only differentiable at $z = 0$. At $z = 0$, $f'(0) = u_x(0,0) + iv_x(0,0) = 0 + i0 = 0$.

Section 24, 4(c): f is already broken into its real and imaginary parts, so $u = e^{-\theta}\cos(\ln r)$ and $v = e^{-\theta}\sin(\ln r)$. So, $u_r = -\frac{1}{r}e^{-\theta}\sin(\ln r)$, $u_\theta = -e^{-\theta}\cos(\ln r)$, $v_r = \frac{1}{r}e^{-\theta}\cos(\ln r)$, and $v_\theta = -e^{-\theta}\sin(\ln r)$.

All four functions are continuous when $r > 0$ and $\theta \in (\alpha, \alpha + 2\pi)$ (the conditions in the problem). (Note that $\ln r$ is NOT continuous at $r = 0$, and that θ is NOT continuous for points with $\theta = \alpha$; there is a "jump" across this ray.) Also, the polar Cauchy-Riemann equations $ru_r = v_\theta$ and $u_\theta = -rv_r$ are satisfied for all r, θ where they exist, so f is analytic on the desired domain. Also,

$$f' = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(-\tfrac{1}{r}e^{-\theta}\sin(\ln r) + \tfrac{1}{r}e^{-\theta}\cos(\ln r)) =$$
$$\tfrac{1}{re^{i\theta}}(-e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r)) = \tfrac{i}{z}(e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)) = \tfrac{i}{z}f(z).$$

Section 26, 1(c): f is already broken into its real and imaginary pieces, so $u = e^{-y}\sin x$ and $v = -e^{-y}\cos x$. So, $u_x = e^{-y}\cos x$, $u_y = -e^{-y}\sin x$, $v_x = -e^{-y}\sin x$, and $v_y = e^{-y}\sin x$. Clearly, all four functions are continuous everywhere, and the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied everywhere in $\mathbb{C}$. So, f is entire.

Section 26, 6: The composite function $G(z) = g(z^2+1)$ can be written as a composite $g(f(z))$, where $f(z) = z^2+1$, and $g(z) = \ln|z| + i\arg_0 z$, or alternately $g(r,\theta) = \ln r + i\theta$, where $\theta \in (0, 2\pi)$.

Firstly, g satisfies the polar Cauchy-Riemann equations everywhere in its cut-plane of analyticity: $ru_r = r(1/r) = 1 = v_\theta$, and $u_\theta = 0 = -rv_r$. So, $g'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(1/r) = (re^{i\theta})^{-1} = 1/z$ as long as z is not 0 or on the positive real axis (it's the positive real axis because we chose the branch $\arg_0$ of arg).

Define the domain D to be the open first quadrant $\{z : x, y > 0\}$. Then to show that $G(z) = g(f(z))$ is analytic on D, we need to show that

(i) f is analytic on D
(ii) g is analytic on $f(D)$

(i) is obvious, since f is a polynomial and therefore entire. To see (ii), let's figure out what $f(D)$ is. The function $f(z) = z^2+1$ can be thought of as first squaring z, then shifting right by 1. We can think of the first quadrant as $\{z : r > 0, \theta \in (0, \pi/2)\}$, and so squaring yields the set $\{z : r > 0, \theta \in (0, \pi)\}$, or the open upper half-plane. Shifting to the right by 1 doesn't change this, so $f(D)$ is the open upper half-plane.

This means that (ii) is true, since g is analytic except at 0 and on the positive real axis; D doesn't contain any of these points, so g is analytic on $f(D)$. Since (i) and (ii) are true, by the chain rule, $G(z)$ is analytic on D.

We can find the derivative of $G(z)$ on D with the chain rule: $G'(z) = g'(f(z))f'(z) = \frac{1}{f(z)} \cdot 2z = \frac{2z}{z^2+1}$.

Extra problem 1: As long as $z \neq 0$, $u_x = \frac{-2xy}{(x^2+y^2)^2}$, $u_{xx} = \frac{6x^2y-2y^3}{(x^2+y^2)^3}$, $u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$, $u_{yy} = \frac{-6x^2y+2y^3}{(x^2+y^2)^3}$, so $u_{xx} + u_{yy} = 0$ on $\{z \neq 0\}$, and u is harmonic on $\{z \neq 0\}$. (Note that there's no hope of u being harmonic at 0, since it doesn't even exist there!)

To find a harmonic conjugate, we need a function v so that $u_x = v_y$ and $u_y = -v_x$. This means that $v_x = \frac{-x^2+y^2}{(x^2+y^2)^2}$ and $v_y = \frac{-2xy}{(x^2+y^2)^2}$. Integrate the second equation with respect to y:

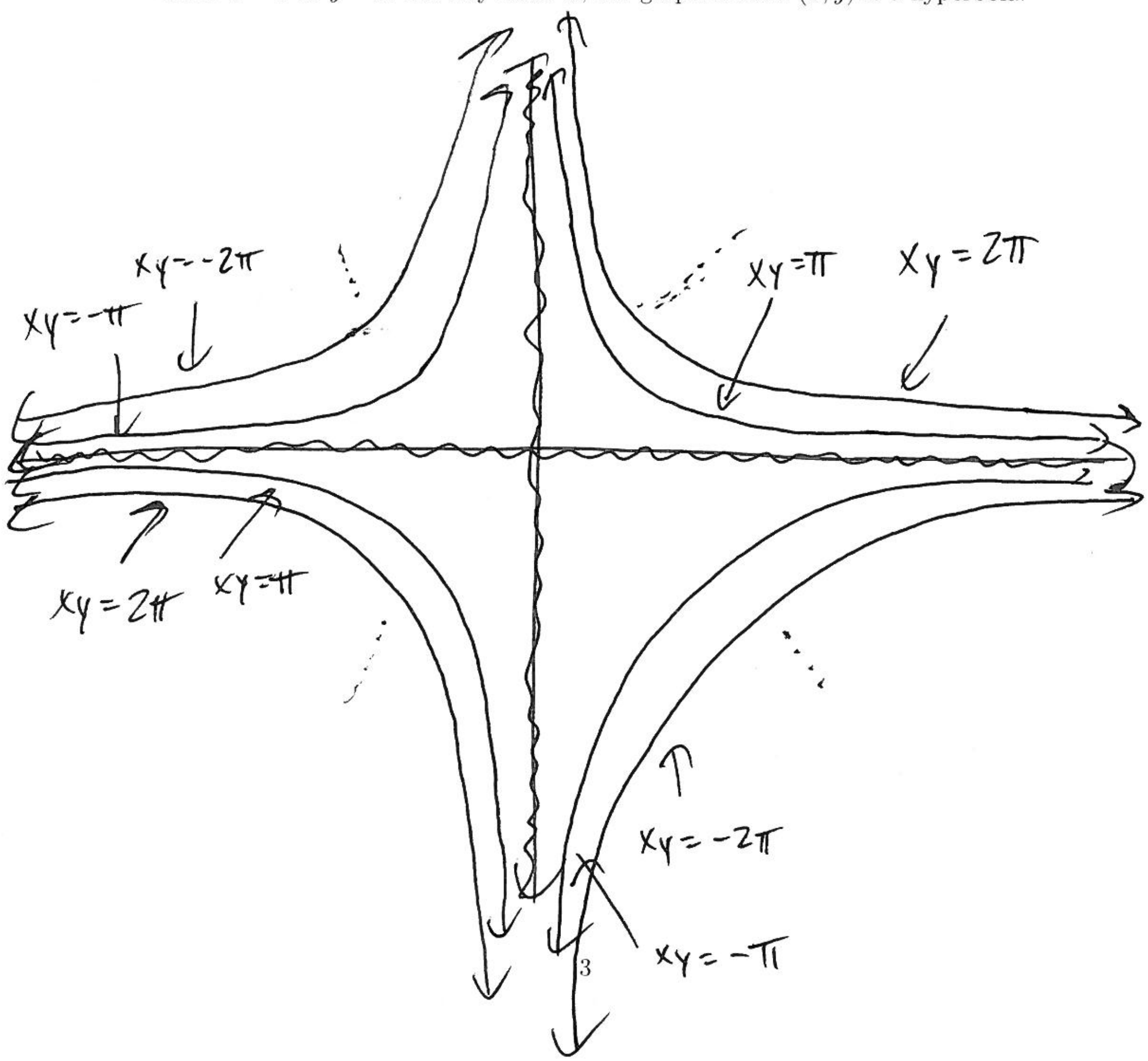
$v = \int \frac{-2xy}{(x^2+y^2)^2}\,dy = \frac{x}{x^2+y^2} + g(x)$. Then $v_x = \frac{-x^2+y^2}{(x^2+y^2)^2} + g'(x)$, so $g'(x) = 0$, meaning that $g(x) = C$ for some constant C. Therefore, $v = \frac{x}{x^2+y^2} + C$ is a harmonic conjugate of u, in the domain $\{z \neq 0\}$.

Interestingly, this could have been noticed another way: $\frac{i}{z} = \frac{i\bar{z}}{|z|^2} = \frac{y+xi}{x^2+y^2} = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2}$ is analytic in the domain $\{z \neq 0\}$, and so it is obvious that $\frac{x}{x^2+y^2}$ is a harmonic conjugate of $\frac{y}{x^2+y^2}$ there.

Extra problem 2: If we define $f = \cos(xy)$, then $u = \cos(xy)$ and $v = 0$. Therefore, $u_x = -y\sin(xy)$, $u_y = -x\sin(xy)$, $v_x = 0$, and $v_y = 0$. These functions are all continuous on all of $\mathbb{C}$, so we just need to check where the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold.

The equations hold if and only if $y\sin(xy) = 0$ and $x\sin(xy) = 0$. This means that either $\sin(xy) = 0$ or both x and y are 0. The second case is contained in the first, so we won't discuss it.

$\sin(xy) = 0$ means that $xy = n\pi$ for some integer n. For $n = 0$, this means that either $x = 0$ or $y = 0$. For any other n, the graph of such (x, y) is a hyperbola:



So, the set of points where f is differentiable is the union of the displayed hyperbolas and the real and imaginary axes. However, this set contains no neighborhoods, so f is not analytic at any point.