Solutions MATH 3851 Homework Assignment 5 (part 2)

Section 30:

12. First, let's write $1/z$ in rectangular form:

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

So,

$$e^{1/z} = e^{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}} = e^{x/(x^2+y^2)}\left(\cos\frac{-y}{x^2+y^2} + i\sin\frac{-y}{x^2+y^2}\right),$$

and

$$\mathrm{Re}(e^{1/z}) = e^{x/(x^2+y^2)}\cos\frac{-y}{x^2+y^2}.$$

Rather than taking partial derivatives to check that this function is harmonic, we just recall that whenever f is analytic in a domain D, its real part is harmonic in D. Also, $e^{1/z}$ is the composition of e^z, which is entire, and $1/z$, which is analytic except at 0. Therefore, $e^{1/z}$ is analytic on $\mathbb{C}\setminus\{0\}$, and so $\mathrm{Re}(1/z)$ is harmonic on $\mathbb{C}\setminus\{0\}$.

Section 33:

2(c).

$$\log(-1+\sqrt{3}i) = \ln|-1+\sqrt{3}i| + i\arg(-1+\sqrt{3}i).$$

Also, $|-1+\sqrt{3}i| = \sqrt{(-1)^2+\sqrt{3}^2} = 2$ and $\arg(-1+\sqrt{3}i) = \frac{2\pi}{3} + 2\pi n$. So,

$$\log(-1+\sqrt{3}i) = \ln 2 + i\left(\frac{2\pi}{3} + 2\pi n\right),$$

where n is any integer.

3.

$$\mathrm{Log}(i^3) = \mathrm{Log}(-i) = \ln|-i| + i\mathrm{Log}(-i) = \ln 1 + i(-\pi/2) = i\frac{-\pi}{2}.$$

On the other hand,

$$3\mathrm{Log}(i) = 3(\ln|i| + i\mathrm{Log}(i)) = 3(\ln 1 + i(\pi/2)) = i\frac{3\pi}{2}.$$

Section 34:

1. Suppose that z_1 and z_2 are complex numbers. Then

$$\mathrm{Log}(z_1z_2) = \ln|z_1z_2| + i\mathrm{Arg}(z_1z_2),$$

and

$$\mathrm{Log}(z_1)+\mathrm{Log}(z_2) = \ln|z_1|+i\mathrm{Arg}(z_1)+\ln|z_2|+i\mathrm{Arg}(z_2) = \ln|z_1z_2|+i(\mathrm{Arg}(z_1)+\mathrm{Arg}(z_2)).$$

We know that $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ is ONE OF THE values of $\arg(z_1z_2)$. But, it may not be the principal value since it may not live in $(-\pi/2, \pi/2)$. On the other hand, each of $\mathrm{Arg}(z_1)$ and $\mathrm{Arg}(z_2)$ is in $(\pi/2, \pi/2)$, so their sum is definitely in $(-\pi, \pi)$. Therefore, if it is not in $(-\pi/2, \pi/2)$, it can be placed inside that interval by just adding or subtracting 2π, either of which gives another legal value for $\arg(z_1z_2)$. Therefore, $\mathrm{Arg}(z_1z_2)$ is $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + 2\pi N$ for either $N = 0$ or $N = \pm 1$. This implies from the above formulas that $\mathrm{Log}(z_1z_2)$ is $\mathrm{Log}(z_1) + \mathrm{Log}(z_2) + i2\pi N$ for either $N = 0$ or $N = \pm 1$.

Section 36:

2(c). The principal value of $(1 - i)^{4i}$ is

$$e^{(4i)\mathrm{Log}(1-i)} = e^{(4i)(\ln|1-i|+i\mathrm{Arg}(1-i))} = e^{(4i)(\ln\sqrt{2}-i(\frac{\pi}{4}))} = e^{\pi+4i\ln\sqrt{2}} = e^{\pi+i\ln 4}$$

$$= e^{\pi}(\cos\ln 4 + i\sin\ln 4) = e^{\pi}\cos\ln 4 + ie^{\pi}\sin\ln 4.$$

7. The values of $i^c = i^{a+bi}$ are given by

$$i^{a+bi} = e^{(a+bi)\log i} = e^{(a+bi)(\ln|i|+\arg i)} = e^{(a+bi)(\frac{\pi}{2}+2n\pi)} = e^{a\frac{\pi}{2}-2bn\pi+i(b\frac{\pi}{2}+2an\pi)}.$$

The modulus of these is given by $e^{a\frac{\pi}{2}-2bn\pi}$. These values will all be different if $b \neq 0$, since e^x is a 1-1 function on the reals, and will obviously all be identical if $b = 0$. So, the moduli of all values of i^c are identical if and only if $c = a + bi$ is real.

Section 38:

11.

$$\sin(\overline{z}) = \sin(x - yi) = \frac{e^{i(x-yi)} - e^{-i(x-yi)}}{2i} = \frac{e^{y+ix} - e^{-y-ix}}{2i}$$

$$= -\frac{i}{2}(e^y(\cos x + i\sin x) - e^{-y}(\cos(-x) + i\sin(x)))$$

$$= -\frac{i}{2}(e^y\cos x+ie^y\sin x-e^{-y}\cos x+ie^{-y}\sin x) = \sin x(\frac{e^y + e^{-y}}{2})+i\cos x(\frac{-e^y + e^{-y}}{2}).$$

Then $u = \sin x(\frac{e^y+e^{-y}}{2})$, so $u_x = \cos x(\frac{e^y+e^{-y}}{2})$ and $u_y = \sin x(\frac{e^y-e^{-y}}{2})$. Similarly, $v = \cos x(\frac{-e^y+e^{-y}}{2})$, so $v_x = -\sin x(\frac{-e^y+e^{-y}}{2})$ and $v_y = \cos x(\frac{-e^y-e^{-y}}{2})$.

The only way for f to be differentiable at a point $z = x + iy$ is if the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ are satisfied there. But here, it's easy to see that the Cauchy-Riemann equations are only satisfied if $\cos x$ and y are both 0, i.e. if $z = \frac{\pi}{2} + 2n\pi$. But there is no neighborhood contained in this set, so $\sin(\overline{z})$ is not analytic anywhere.

The proof for $\cos(\overline{z})$ is trivially similar.

14(b). Suppose that $\overline{\sin(iz)} = \sin(i\overline{z})$. Then,

$$\overline{\left(\frac{e^{i(i(x+iy))} - e^{-i(i(x+iy))}}{2i}\right)} = \frac{e^{i(i(x-iy))} - e^{-i(i(x-iy))}}{2i}, \text{ and so}$$

$$\overline{-\frac{i}{2}(e^{-x-iy} - e^{x+iy})} = -\frac{i}{2}(e^{-x+iy} - e^{x-iy}), \text{ and so}$$

$$\overline{-\frac{i}{2}(e^{-x}(\cos(-y) + i\sin(-y)) - e^{x}(\cos y + i\sin y))}$$
$$= -\frac{i}{2}(e^{-x}(\cos y + i\sin y) - e^{x}(\cos(-y) + i\sin(-y))), \text{ and so}$$

$$\overline{\frac{-e^{-x} - e^{x}}{2}\sin y + i\frac{-e^{-x} + e^{x}}{2}\cos y} = \frac{e^{-x} + e^{x}}{2}\sin y + i\frac{-e^{-x} + e^{x}}{2}\cos y, \text{ and so}$$
$$\frac{-e^{-x} - e^{x}}{2}\sin y - i\frac{-e^{-x} + e^{x}}{2}\cos y = \frac{e^{-x} + e^{x}}{2}\sin y + i\frac{-e^{-x} + e^{x}}{2}\cos y.$$

Matching the real parts shows that either $e^x + e^{-x} = 0$ (impossible) or $\sin y = 0$. Therefore, $y = n\pi$ for some integer n. Matching the imaginary parts shows that either $\cos y = 0$ (impossible since $y = n\pi$) or $e^x - e^{-x} = 0$, meaning that $x = 0$. Therefore, $x = 0$ and $y = n\pi$, and so $z = n\pi i$ for some integer n.

Section 42:

2(a).

$$\int_0^1 (1+it)^2\,dt = \frac{(1+it)^3}{3i}\Big|_0^1 = \frac{(1+i)^3}{3i} - \frac{1}{3i} = \frac{1+3i+3i^2+i^3-1}{3i} = \frac{-3+2i}{3i} = \frac{2}{3} + i.$$

Extra problem 1: If $\sin z = i$, then $\frac{e^{iz} - e^{-iz}}{2i} = i$, so $e^{iz} - e^{-iz} = -2$. We substitute $w = e^{iz}$, and get

$$w - w^{-1} = -2 \Longrightarrow w^2 - 1 = -2w \Longrightarrow w^2 + 2w - 1 = 0.$$

Therefore, by the quadratic formula, $w = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$. We can then solve for z in each case:

$$w = -1 + \sqrt{2} \Longrightarrow iz = \log(-1+\sqrt{2}) \Longrightarrow z = -i\log(-1+\sqrt{2})$$
$$= -i\left(\ln|-1+\sqrt{2}| + i\arg(-1+\sqrt{2})\right) = -i(\ln(\sqrt{2}-1) + i2n\pi) = 2n\pi - i\ln(\sqrt{2}-1).$$

Similarly,

$$w = -1-\sqrt{2} \Longrightarrow iz = \log(-1-\sqrt{2}) \Longrightarrow z = -i\log(-1-\sqrt{2})$$
$$= -i\left(\ln|-1-\sqrt{2}| + i\arg(-1-\sqrt{2})\right) = -i(\ln(\sqrt{2}+1)+i(\pi+2n\pi)) = (\pi+2n\pi)-i\ln(\sqrt{2}+1).$$

Extra problem 2: We have a composite function $f(g(z))$, where $f(z)$ is a branch of $\log z$ and $g(z) = iz^2$. In order for the composition $f(g(z))$ to be analytic on a domain D, we need two conditions:

1. $g(z)$ must be analytic on D. This is obvious; iz^2 is an entire function, so clearly it is analytic on D specifically.
2. $f(z)$ must be analytic on $g(D)$, the image of D under $g(z)$. For this, we must figure out what $g(D)$ is.

Remember that $D = \{z : y > 0\}$, which is clearly the same as $\{z : \arg(z) \in (0,\pi)\}$. The effect of iz^2 on a complex number z is to double its argument and then rotate counterclockwise by $\pi/2$ radians (i.e. to add $\pi/2$ to its argument). Therefore, $g(D) = \{z : \arg(z) \in (\pi/2, 5\pi/2)\}$. This is almost the entire complex plane, except for the nonnegative imaginary axis. This leaves just enough room to choose $f = \log_{\pi/2} z$; then the bad ray for f is the nonnegative imaginary axis, which does not intersect $g(D)$, and so f is analytic on $g(D)$.

Therefore, the branch $\log_{\pi/2}(iz^2)$ is analytic on D, and so is a solution (and the only one!) to this problem.