Math 3851 Midterm Exam

Instructions: Please answer all questions completely and show your work unless otherwise indicated. You may use any results or theorems that we discussed in class without proof, but if you use a result, please clearly state what it is. You may not use textbooks, notes, calculators, or other outside aids for this exam. The exam is worth 100 points and designed to take 90 minutes, but you may use the entire class period for the exam. Good luck!

1. (6 pts. each) Write all values of each of the following expressions in rectangular form (i.e. $z = x + iy$):

(a) $\dfrac{1-i}{1-2i}$

Solution:

$$\frac{1-i}{1-2i} = \frac{1-i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{(1-i)(1+2i)}{(1-2i)(1+2i)} = \frac{1-i+2i+2}{5} = \frac{3}{5} + i\frac{1}{5}.$$

(b) $(\sqrt{2} + \sqrt{2}i)^{100}$

Solution: First we write $\sqrt{2} + i\sqrt{2}$ in polar. The modulus (r) is $|\sqrt{2} + i\sqrt{2}| = \sqrt{\sqrt{2}^2 + \sqrt{2}^2} = \sqrt{4} = 2$, and a value for the argument is $\frac{\pi}{4}$. So, $\sqrt{2} + \sqrt{2}i = 2e^{i(\pi/4)}$. This means that $(\sqrt{2} + \sqrt{2}i)^{100} =$

$$(2e^{i(\pi/4)})^{100} = 2^{100}e^{i(\pi/4)100} = 2^{100}e^{i(25\pi)} = 2^{100}(\cos(25\pi) + i\sin(25\pi)) = -2^{100}.$$

(c) i^i

Solution: By definition, $i^i = e^{i \log i}$, where $\log i$ can take any of its legal values. Also by definition, $\log i = \ln |i| + i\arg(i) = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(\frac{\pi}{2} + 2n\pi\right)$. So,

$$i^i = e^{i \log i} = e^{i\left(i\left(\frac{\pi}{2} + 2n\pi\right)\right)} = e^{-\left(\frac{\pi}{2} + 2n\pi\right)}.$$

2. **(a)** (15 pts.) If $u(x, y) = x^3 + 2xy - 3xy^2$, find $v(x, y)$ so that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is entire. Make sure to explain why your function is entire!

Solution: We need Cauchy-Riemann to hold. Here, $u_x = 3x^2 + 2y - 3y^2$ and $u_y = 2x - 6xy$. We start by setting

$$v_x = -u_y \Longrightarrow v_x = 6xy - 2x \Longrightarrow v = \int 6xy - 2x \, dx = 3x^2y - x^2 + g(y).$$

Then,

$$v_y = u_x \Longrightarrow 3x^2 + g'(y) = 3x^2 + 2y - 3y^2 \Longrightarrow g'(y) = 2y - 3y^2 \Longrightarrow g(y) = \int 2y - 3y^2 \, dy = y^2 - y^3 + C.$$

So, it looks like $v = 3x^2y - x^3 + y^2 - y^3 + C$.

We know that the Cauchy-Riemann equations hold on all of $\mathbb{C}$ because we just checked them, and all partial derivatives are polynomials of x and y, therefore they are continuous on $\mathbb{C}$. Therefore, $f = u + iv$ is analytic on all of $\mathbb{C}$, and so entire.

(b) (5 pts.) Explain why it is impossible to find such a $v(x, y)$ if we instead chose $u(x, y) = x^3 + 2xy - 4xy^2$.

Solution: We know that if $f = u + iv$ is analytic on $\mathbb{C}$, then u must be harmonic on $\mathbb{C}$. However, if we check where u is harmonic, we get $u_{xx} = 8x$ and $u_{yy} = -6x$. Therefore, $u_{xx} + u_{yy} = 2x$, which is 0 only if $x = 0$. So, u is harmonic only when $x = 0$, and therefore not harmonic on $\mathbb{C}$, and so there can be no v for which $u + iv$ is entire.

3. (12 pts.) Describe all solutions to

$$e^z = e^{iz} \tag{1}$$

in rectangular form, i.e. give conditions on x and y which are equivalent to $z = x + iy$ satisfying (1).

Solution:

$$e^z = e^{iz} \iff e^{x+iy} = e^{i(x+iy)} \iff e^{x+iy} = e{-y+ix} \iff e^x e^{iy} = e^{-y} e^{ix}.$$

But both $e^x e^{iy}$ and $e^{-y} e^{ix}$ are written in polar form. So, they are the same number if and only if $e^x = e^{-y}$ and $e^{iy} = e^{ix}$. By taking natural logs, we see $e^x = e^{-y} \iff x = -y$. We can then rewrite the second equation:

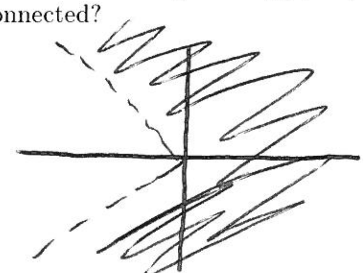
$$e^{iy} = e^{ix} \iff e^{iy} = e^{-iy} \iff e^{i2y} = 1 \iff \cos(2y) + i\sin(2y) = 1.$$

This is true if and only if $\cos(2y) = 1$ and $\sin(2y) = 0$, which is true if and only if $2y = n\pi$ for some integer n.

So, finally we see that $e^z = e^{iz} \iff y = n(\pi/2), x = -n(\pi/2)$ for some integer n.

4. (a) (3 pts.) Sketch the set $S = \{z : \mathrm{Arg}(z) \in (-\frac{3\pi}{4}, \frac{3\pi}{4})\}$. Is S a domain, i.e. is S open and connected?

Solution:

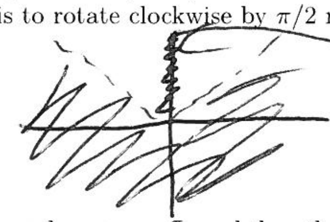


yes, S is a domain

(b) (15 pts.) Find a branch $f(z)$ of the multiple-valued expression $\log(-iz)$ which is analytic for all z in S.

Solution: First we write $\log(-iz)$ as a composition $f(g(z))$, where $g(z) = -iz$ and $f(z)$ is some branch of $\log z$. Then $f(g(z))$ is analytic in S if g is analytic in S and f is analytic in $g(S)$. Since g is a polynomial, it is entire and therefore definitely analytic in S.

We only need to choose a branch of $\log z$ which is analytic in $g(S)$. Since g just multiplies its input by $-i$, and we can write $-i$ in polar as $e^{i(-\pi/2)}$, the effect of g is to rotate clockwise by $\pi/2$ radiance. Therefore, $g(S)$ looks like



bad ray for $\log_{\pi/2} z$

So, we can choose $\alpha = \frac{\pi}{2}$, and then the branch $\log_\alpha z$ is analytic in $g(S)$ since its "bad ray" does not touch $g(S)$. Therefore, $\log_\alpha(-iz)$ is analytic throughout S.

(c) (3 pts.) For the branch you chose in part (b), express $f(2i)$ in rectangular form. (In other words, "plug in" $z = 2i$ to $\log(-iz)$ for the branch you chose.)

Solution: $\log_{\pi/2}(-i(2i)) = \log_{\pi/2}(2) = \ln|2| + i\arg_{\pi/2}(2) = \ln 2 + i(2\pi)$. (All values of $\arg(2)$ are $0, 2\pi, -2\pi, 4\pi, \ldots$, and $\arg_{\pi/2}$ is the one of those in the interval $(\pi/2, \pi/2 + 2\pi)$, which is 2π.)

5. (10 pts.) If $|z| = 4$, show that $|\overline{z}^3 - 2i(z+1)^2| \geq 14$. (Hint: start by saying something about $|\overline{z}|$ and $|z+1|$.)

Solution: Since $|z| = 4$, $|\overline{z}| = |z| = 4$. Also, the triangle and reverse triangle inequalities show that $|z| - |1| \leq |z+1| \leq |z| + |1|$, i.e. $3 \leq |z+1| \leq 5$. Then, by the reverse triangle inequality,

$$|\overline{z}^3 - 2i(z+1)^2| \geq |\overline{z}^3| - |2i(z+1)^2| = |\overline{z}|^3 - |2i||z+1|^2 = 64 - 2|z+1|^2.$$

Since $|z+1| \leq 5$, $64 - 2|z+1|^2 \geq 64 - 2 \cdot 5^2 = 14$, so we've shown that $|\overline{z}^3 - 2i(z+1)^2| \geq 14$.

6. (12 pts.) Show that the function $f(z) = \dfrac{\sin \overline{z}}{\sin z}$ DOES NOT have a limit as $z \to 0$. (Reminder: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$.)

Solution: If we let $z \to 0$ along the path $z = x$ (where $x \to 0$, then we get

$$\lim_{x \to 0} \frac{\sin \overline{x}}{\sin x} = \lim_{x \to 0} \frac{\sin x}{\sin x} = 1.$$

If we let $z \to 0$ along the path $z = iy$ (where $y \to 0$, then we get

$$\lim_{y \to 0} \frac{\sin \overline{iy}}{\sin iy} = \lim_{y \to 0} \frac{\sin(-iy)}{\sin(iy)} = \lim_{y \to 0} \frac{\frac{e^{y} - e^{-y}}{2i}}{\frac{e^{-y} - e^{y}}{2i}} = -1.$$

Since $f(z)$ approaches different limits as $z \to 0$ along two different paths, $\lim_{z \to 0} \frac{\sin \overline{z}}{\sin z}$ does not exist.

7. (7 pts.) Explain why the function $f(z) = \left|e^{e^{e^z}}\right|$ is NOT entire, i.e. analytic throughout $\mathbb{C}$. (Hint: computing partial derivatives for Cauchy-Riemann here looks pretty terrible; do you notice anything about this function that would let you conclude non-analyticity more easily?)

Solution: Since it is the modulus of a complex number, f is real-valued throughout $\mathbb{C}$. We showed in class that if f is real-valued and entire, then it must be constant. Since $f(z)$ is definitely not a constant, it cannot possibly be entire.