Selected solutions for MATH 4280 Assignment 1

1.11.(ii) The forward direction is obvious and was proved in class. We therefore need prove only the following: if μ is finitely additive, $\mu(X) < \infty$, and μ is continuous from above, then μ is a measure. It suffices to prove countable additivity, since we already know that μ takes values in $[0, \infty]$ and that $\mu(\varnothing) = 0$.

For this, consider any disjoint measurable sets $A_1, A_2, \ldots \in \mathcal{M}$. Define the unions $B_n = \bigcup_{i=1}^{n} A_i$; then $B_1 \subseteq B_2 \subseteq \ldots$, and so clearly $B_1^c \supseteq B_2^c \supseteq \ldots$. Also, all B_n^c are in $\mathcal{M}$ since it is a σ-algebra. Therefore, by continuity from above,

$$\mu\left(\bigcap_{n=1}^{\infty} B_n^c\right) = \lim_{n\to\infty} \mu(B_n^c).$$

Using de Morgan's Law yields

$$\mu\left(\left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) = \lim_{n\to\infty} \mu(B_n^c).$$

Since μ is finitely additive, for any set $C \in \mathcal{M}$, $\mu(C) + \mu(C^c) = \mu(X)$, and so $\mu(C^c) = \mu(X) - \mu(C)$. We therefore get

$$\mu(X) - \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(X) - \mu(B_n),$$

implying (since $\mu(X)$ is finite!) that

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n).$$

Note that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, and that by finite additivity, for all n we have $\mu(B_n) = \sum_{i=1}^{n} \mu(A_i)$. Therefore,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i),$$

completing the proof of countable additivity and the proof that μ is a measure.

1.14. Suppose that μ is semifinite and that $E \in \mathcal{M}$ with $\mu(E) = \infty$. Consider the set

$$S = \{\mu(F) \ : \ F \in \mathcal{M}, \mu(F) < \infty, F \subseteq E\}.$$

Then S is nonempty since μ is semifinite. Let's assume for a contradiction that S is bounded from above. Then, define $\alpha = \sup S$. By definition of supremum, for every n there exists $F_n \in \mathcal{M}$, $F_n \subseteq E$, with $\alpha - \frac{1}{n} < \mu(F_n)$. Define

$F' = \bigcup_{n\in\mathbb{N}} F_n$. Then clearly $F' \in \mathcal{M}$ and $F' \subseteq E$. Also, by monotonicity, for every n $\mu(F') \geq \mu(F_n) > \alpha - \frac{1}{n}$, and so $\mu(F') \geq \alpha$.

We claim that $\mu(F') = \alpha$, so let's assume for a contradiction that $\mu(F') > \alpha$. Then, by continuity from below, $\lim_{n\to\infty} \mu\left(\bigcup_{i=1}^n F_i\right) = \mu(F') > \alpha$. This means that there exists n so that $\mu\left(\bigcup_{i=1}^n F_i\right) = \mu(F') > \alpha$, and $\mu\left(\bigcup_{i=1}^n F_i\right)$ is finite since each $\mu(F_i)$ is finite. We've therefore contradicted the definition of α as $\sup S$.

We now know that $\mu(F') = \alpha$. Then, by finite additivity, $\mu(E \setminus F') = \infty$. Therefore, since μ is semifinite, there exists $G \in \mathcal{M}$, $G \subset E \setminus F'$ with $0 < \mu(G) < \infty$. Finally, consider the set $F' \cup G$. It is in $\mathcal{M}$ since F' and G are, and similarly it is a subset of E. Also, $\mu(F' \cup G) = \mu(F') + \mu(G)$ since F' and G are disjoint, and $\alpha = \mu(F') < \mu(F') + \mu(G) < \infty$. We've again contradicted the definition of α as the supremum of S.

Therefore, our original assumption was wrong, and S is unbounded from above. But then for every $C > 0$ there exists $F \subseteq E$ with $C < \mu(F) < \infty$, and so we are done.