Selected solutions for MATH 4280 Assignment 2

1.18(c). The proof of the reverse direction from part (b) should not have used the hypothesis that $\mu^*(E) < \infty$, so we need only show that the forward direction can substitute σ-finiteness of μ_0 as a hypothesis. So, assume that μ_0 is σ-finite and that E is μ^*-measurable.

Since μ_0 is σ-finite, there exist $F_1, F_2, \ldots \in \mathcal{A}$ so that for all n, $\mu_0(F_k) < \infty$, and $X = \bigcup_{k=1}^{\infty} F_k$. We can assume without loss of generality that the F_k are disjoint by the fact that $\mathcal{A}$ is an algebra and monotonicity (via finite additivity) of μ_0. Also, since each F_k is in $\mathcal{A}$, $\mu^*(F_k) = \mu_0(F_k) < \infty$ for every k.

Now, for every n, define $E_k = E \cap F_k$; by monotonicity, $\mu^*(E_k) \leq \mu^*(F_k) < \infty$. Then by part (a), choose $A_{n,k} \in \mathcal{A}_\sigma$ so that $E_k \subseteq A_{n,k}$ and $\mu^*(A_{n,k}) < \mu^*(E_k) + \frac{1}{n2^{|k|}}$. Since E is μ^*-measurable,

$$\mu^*(A_{n,k}) = \mu^*(A_{n,k} \cap E_k) + \mu^*(A_{n,k} \cap E_k^c) = \mu^*(E_k) + \mu^*(A_{n,k} \setminus E_k),$$

so since all of these values are finite, $\mu^*(A_{n,k} \setminus E_k) < \frac{1}{n2^{|k|}}$.

Then, for every n, define $A_n = \bigcup_{k\in\mathbb{Z}} A_{n,k}$ and $A = \bigcap_{n\in\mathbb{Z}} A_n$. Then, clearly $A \in \mathcal{A}_{\sigma\delta}$, $E \subseteq A$, and and $A \setminus E \subseteq A_n \setminus E$. So,by monotonicity and countable subadditivity,

$$\mu^*(A \setminus E) \leq \mu^*(A_n \setminus E) \leq \sum_{k\in\mathbb{Z}} \mu^*(A_{n,k} \setminus E_k) < \sum_{k\in\mathbb{Z}} \frac{1}{n2^{|k|}} = \frac{3}{n}.$$

Since n was arbitrary, $\mu^*(A \setminus E) = 0$ and we're done.

1.23(b). Clearly $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{P}(\mathbb{Q})$ since $A \subset \mathcal{P}(\mathbb{Q})$. Conversely, for every $q \in \mathbb{Q}$, $\{q\} = \bigcap_{n\in\mathbb{N}}((q-n^{-1}, q+n^{-1}] \cap \mathbb{Q})$, and so $\{q\} \in \mathcal{M}(\mathcal{A})$. Then, by Lemma 1.1, $\mathcal{M}(\{\{q\}_{q\in\mathbb{Q}}\}) = \mathcal{P}(\mathbb{Q}) \subseteq \mathcal{M}(\mathcal{A})$, and so the sets are equal.

(c) Define the following two measures on $\mathcal{P}(\mathbb{Q})$. μ_1 is defined by $\mu_1(A) = 0$ if A is empty, and $\mu_1(A) = \infty$ if A is nonempty. μ_2 is defined by $\mu_2(A) = |A|$, i.e. the counting measure on $\mathbb{Q}$. It is simple to check that both are measures, and both extend μ_0 since the only nonempty sets in $\mathcal{A}$ were infinite.

We note that the only reason that μ_0 did not have a unique extension to $\mathcal{M}(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$ is that μ_0 was not σ-finite; the only set with finite μ_0 was the empty set.

Written problem:

$\Longleftarrow$: Suppose that $F(x) = ax+b$, and consider μ_F induced in the usual way. Then for any interval $I = (c,d]$, $\mu_F(I) = F(d) - F(c) = (d-c)x$, depending only on the length of I and meaning that $\mu_F(I) = \mu_F(I+x)$ for any $x \in \mathbb{R}$.

Now, for any fixed interval $I = (\alpha, \beta]$, consider the collection $\mathcal{C}_I = \{A \subseteq I : \forall x \in \mathbb{R}, \mu_F(A) = \mu_F((A + x)\}$ of subsets of I for which μ_F is shift-invariant.

By the above, $\mathcal{C}_I$ contains all subintervals of I of the form $(c, d]$. We claim that $\mathcal{C}_I$ is a σ-algebra on I. First, consider any $A \in \mathcal{C}$. Then for all $x \in \mathbb{R}$,

$$\mu_F((I\backslash A)+x) = \mu_F((I+x)\backslash(A+x)) = \mu_F(I+x)-\mu_F(A+x) = \mu_F(I)-\mu_F(A) = \mu_F(I\backslash A).$$

(We here used the fact that μ_F is finite on the bounded set $I + x$.) So, $\mathcal{C}_I$ is closed under complements (viewing I as the universal space.) Similarly, for any disjoint collection $A_1, A_2, \ldots \in \mathcal{C}_I$,

$$\mu_F\left(x + \bigcup_{n=1}^{\infty} A_n\right) = \mu_F\left(\bigcup_{n=1}^{\infty}(x + A_n)\right) = \sum_{n=1}^{\infty} \mu_F(x+A_n) = \sum_{n=1}^{\infty} \mu_F(A_n) = \mu_F\left(\bigcup_{n=1}^{\infty} A_n\right).$$

So, $\mathcal{C}_I$ is also closed under countable unions, and so is a σ-algebra over I. Since it also contains all half-open subintervals of I, in fact $\mathcal{C}_I \supseteq \mathcal{B}(I)$.

Then, for any Borel subset E of $\mathbb{R}$, each of the sets $E_n = E \cap [n, n + 1)$ is a Borel subset of $[n, n + 1)$, and so by the above is in $\mathcal{C}_{[n,n+1)}$. Then, for any x,

$$\mu(E + x) = \sum_{n=1}^{\infty} \mu(E_n + x) = \sum_{n=1}^{\infty} \mu(E_n) = \mu(E),$$

completing the proof.

$\Longrightarrow$: Suppose that μ_F is shift-invariant. Then, $\mu_F((c, d]) = \mu_F((c+x, d+x])$ for all $c < d$ and x, so $F(d) - F(c) = F(d + x) - F(c + x)$ for all $c < d$ and x. Define $G(x) = F(x) - F(0)$; then $G(0) = 0$ and G also has the property that $G(d) - G(c) = G(d + x) - G(c + x)$ for all $c < d$ and x, and G is also right-continuous. In particular, for any natural n,

$$G(n) = G(n) - G(0) = \sum_{i=0}^{n-1} G(i + 1) - G(i) = n(G(1) - G(0)) = nG(1).$$

Finally, for any natural numbers m, n,

$$G(n) = G(n)-G(0) = \sum_{i=0}^{m-1} G((i+1)n/m)-G(in/m) = m(G(n/m)-G(0)) = mG(n/m).$$

In other words, $G(n/m) = (n/m)G(1)$. A similar argument works for negative n, so in fact for all $q \in \mathbb{Q}$, $G(q) = qG(1)$. Finally, for any real α, there exists a sequence of rationals q_n approaching α from above, and so by right-continuity of G,

$$G(\alpha) = \lim_{n\to\infty} G(q_n) = \lim_{n\to\infty} q_n G(1) = \alpha G(1).$$

So, $G(x) = G(1)x$ for all x, implying that $F(x) = (F(1) - F(0))x + F(0)$ for all x and completing the proof.