Selected solutions for MATH 4280 Assignment 3

1.30. Suppose that $m(E) > 0$, and choose any $\epsilon > 0$. First of all, m is σ-finite, thus semi-finite. Therefore, if $m(E) = \infty$, we can pass to a subset F with $0 < m(F) < \infty$, and if we show that $m(I \cap F) > (1-\epsilon)m(I)$ for some interval I, then of course the same is true of E, since by monotonicity $m(I \cap E) \geq m(I \cap F)$. So, without loss of generality, we assume that $0 < m(E) < \infty$.

Choose δ so that $\frac{\delta}{m(E)-\delta} < \epsilon$. Then, by Littlewood's first principle, we can choose disjoint intervals $I_1, \ldots, I_n$ so that if we denote their union by I, then $m(E \triangle I) < \delta$. In particular, this implies that $m(I) \geq m(E \cap I) = m(E) - m(E \setminus I) \geq m(E) - m(E \triangle I) > m(E) - \delta$.

Then, $m(E \cap I) = m(I) - m(I \setminus E) \geq m(I) - m(E \triangle I) > m(I) - \delta = m(I)(1 - \frac{\delta}{m(I)}) = m(I)(1 - \frac{\delta}{m(E)-\delta}) > m(I)(1-\epsilon)$. In other words, E "covers proportion of more than $1-\epsilon$" of the finite union of intervals I. We now must show that this property passes to some individual interval.

If every one of the intervals I_i had the property that $m(E \cap I_i) \leq m(I_i)(1-\epsilon)$, then we would have

$$m(E \cap I) = \sum_{i=1}^{n} m(E \cap I_i) \leq \sum_{i=1}^{n} m(I_i)(1-\epsilon) = (1-\epsilon)\sum_{i=1}^{n} m(I_i) = m(I)(1-\epsilon),$$

a contradiction. Therefore, one of the intervals I_i satisfies $m(E \cap I_i) > m(I_i)(1-\epsilon)$, and so we are done.

1.31. Consider any set E with $m(E) > 0$. By problem 30, there exists an interval $I = (a,b)$ with $m(E \cap I) > 0.99m(I)$. Now, choose any nonnegative $x \leq 0.98m(I) = 0.98(b-a)$. Then, the sets $E \cap I$ and $(E \cap I) + x$ are both contained in $J = I \cup I + x = (a, b+x)$, and $m(J) = m(I) + x \leq 1.98m(I)$. Also, by shift-invariance of Lebesgue measure, $m(E \cap I) = m((E \cap I) + x) > 0.99m(I)$. If $E \cap I$ and $(E \cap I) + x$ were disjoint, then by finite additivity and monotonicity we would have

$$m(E \cap I) + m((E \cap I) + x) \leq m(J) \leq 1.98m(I),$$

a contradiction. Therefore, $E \cap I$ and $(E \cap I) + x$ are not disjoint, and in particular there exists $y \in E \cap (E + x)$. This means that there exist $e, e' \in E$ where $y = e = e' + x$, and so $x = e' - e \in E - E$. Since x was an arbitrary element of $[0, 0.98m(I)]$, this means that $[0, 0.98m(I)] \subseteq E - E$. The proof for $x \in [-0.98m(I), 0]$ is extremely similar, and we omit details here, but after that verification we have $[-0.98m(I), 0.98m(I)] \subseteq E - E$ and are done.

1.32. (a) We first note that $\log\left(\prod_{j \in \mathbb{N}}(1-\alpha_j)\right) = \sum_{j \in \mathbb{N}} \log(1-\alpha_j)$, and so the problem asked is equivalent to showing that

$$\sum_{j \in \mathbb{N}} a_j < \infty \Longleftrightarrow \sum_{j \in \mathbb{N}} -\log(1-\alpha_j) < \infty.$$

We now note that

$$\lim_{x \to 0^+} \frac{-\log(1-x)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{1-x}}{1} = 1$$

by L'Hospital's Rule.

But now, if we assume either of the series above converges, then clearly $\alpha_j \to 0$ as $j \to \infty$, which implies by the Limit Comparison Test that the other series converges as well, and we're finished.

(Alternately: $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$, which is clearly greater than x, and so $\sum_1^\infty -\log(1-\alpha_j) < \infty \implies \sum_1^\infty \alpha_j < \infty$ by the Comparison Test. For the other direction,

$$-\log(1-x) = x + x^2/2 + x^3/3 + \ldots < x + x^2 + x^3 + \ldots = \frac{x}{1-x},$$

which is less than $2x$ as long as $x < \frac{1}{2}$. Then, if $\sum_1^\infty \alpha_j < \infty$, clearly there exists J so that $\alpha_j < \frac{1}{2}$ for $j > J$. Then, since $\sum_J^\infty 2\alpha_j < \infty$, $\sum_J^\infty -\log(1-\alpha_j) < \infty$ by the Comparison Test, implying that $\sum_1^\infty -\log(1-\alpha_j) < \infty$ as well.)

(b) Choose any $\beta > 0$. The sequence $\beta^{1/2}$, $\beta^{3/4}$, $\beta^{7/8}$, $\ldots$ decreases to β and takes values in $(0,1)$. Therefore, we should be able to choose α_i so that for every n,

$$\prod_{j=1}^{n} (1-\alpha_j) = \beta^{1-2^{-n}}.$$

Indeed, for $n = 1$ we have $1 - \alpha_1 = \beta^{1/2} \implies \alpha_1 = 1 - \beta^{1/2}$, and for $n > 1$ we have

$$1 - \alpha_n = \frac{\prod_{j=1}^{n}(1-\alpha_j)}{\prod_{j=1}^{n-1}(1-\alpha_j)} = \frac{\beta^{1-2^{-n}}}{\beta^{1-2^{-(n-1)}}} = \beta^{2^{-n}}.$$

So, taking $\alpha_n = 1 - \beta^{2^{-n}}$ for all n will yield

$$\prod_{j=1}^{\infty} (1-\alpha_j) = \lim_{n \to \infty} \beta^{1-2^{-n}} = \beta.$$