Selected solutions for MATH 4280 Assignment 5

16. Suppose that $f \in L^+(X, \mathcal{M})$ and that $\int f\, d\mu < \infty$. Then, for each n, define the set $E_n = \{x : f(x) > n\} = f^{-1}((n, \infty))$ and the function $f_n = f\chi_{E_n}$. Then, $E = \bigcup_{n=1}^{\infty} E_n = \{x : f(x) < \infty\}$, and we know by a previous exercise that $\mu(E^c) = 0$. Therefore, the pointwise limit of f_n is $f\chi_E$, which equals f μ-a.e. Also, since $E_1 \subseteq E_2 \subseteq \ldots$, the functions f_n are increasing in n, and so we can apply the Monotone Convergence Theorem to see that

$$\int f_n\, d\mu \to \int f d\mu.$$

Then by definition of convergence, for every ϵ, there exists n so that

$$\int f_n\, d\mu = \int_{E_n} f\, d\mu > \int f\, d\mu - \epsilon.$$

Finally, we note that $n\chi_{E_n} \leq f$, and so by monotonicity,

$$\int n\chi_{E_n}\, d\mu = n\mu(E_n) \leq \int f\, d\mu,$$

implying that $\mu(E_n) < \infty$ and completing the proof.

38(b). "Normal" solution: Suppose $f_n \to f$ in measure and that $g_n \to g$ in measure. We'll also assume that all functions are finite μ-a.e.; the proof is just slightly more technical if we allow them to take infinite values. We fix $\epsilon > 0$. First, we note that

$$|f_n g_n - fg| \leq |f_n g_n - f g_n| + |f g_n - fg| = |g_n||f_n - f| + |f||g_n - g|.$$

Therefore,

$$\{x : |f_n g_n - fg| > \epsilon\} \subset \{x : |g_n||f_n - f| > 0.5\epsilon\} \cup \{x : |f||g_n - g| > 0.5\epsilon\}.$$

We can then, for any $M > 0$, break these sets down further:

$$\{x : |g_n||f_n - f| > 0.5\epsilon\} \subset \{x : |g_n| > M\} \cup \{x : |f_n - f| > 0.5M^{-1}\epsilon\}$$

and

$$\{x : |g_n||f_n - f| > 0.5\epsilon\} \subset \{x : |f| > M\} \cup \{x : |g_n - g| > 0.5M^{-1}\epsilon\}.$$

Our plan of attack is now to take M so large that the sets on the left have very small measure, and then to take n so large (dependent on M) so that the sets on the right have very small measure. However, there is a problem: in theory, our M which gives $\{x : |g_n| > M\}$ small measure depends on n, which would cause circular dependencies.

Note that the sets $G_n := \{x : |g| > n\}$ are decreasing, and their intersection is $\{x : |g| = \infty\}$, a null set. Therefore, for any $\delta > 0$, there exists M' so that $\mu(G_{M'-1}) < 0.125\delta$. But, then by convergence in measure of the functions g_n, there exists N' so that for any $n > N'$, $\mu(\{x : |g_n - g| > 1\}) < 0.125\delta$. We now note that

$$\{x : |g_n| > M'\} \subset \{x : |g| > M' - 1\} \cup \{x : |g - g_n| > 1\},$$

and so for ALL $n > N'$, $\mu(x : |g_n| > M') < 0.125\delta + 0.125\delta = 0.25\delta$. Similarly, choose M'' so that $\mu(\{x : |f| > M''\}) < 0.25\delta$. Then, take $M = \max(M', M'')$. By the fact that $f_n \to f$ in measure, there exists N'' so that $n > N'' \Rightarrow \mu(\{x : |f_n - f| > 0.5M^{-1}\epsilon) < 0.25\delta$. Similarly, there exists N''' so that $n > N''' \Rightarrow \mu(\{x : |g_n - g| > 0.5M^{-1}\epsilon) < 0.25\delta$. Take $N = \max(N', N'', N''')$. Now, finally, we see that for any $n > N$,

$$\begin{aligned}\mu(\{x : |f_n g_n - fg| > \epsilon\}) &\leq \mu(\{x : |g_n| > M\}) + \mu(\{x : |f_n - f| > 0.5M^{-1}\epsilon\}) \\ &\quad + \mu(\{x : |f| > M\}) + \mu(\{x : |g_n - g| > 0.5M^{-1}\epsilon\}) = \delta.\end{aligned}$$

Since $\delta > 0$ is arbitrary, this shows that $f_n g_n \to fg$ in measure.

Terry's really cool solution: Choose any increasing sequence n_k of natural numbers. Since $\{f_n\}$ converges to f in measure, the same is true of $\{f_{n_k}\}$. Therefore, there is a subsequence $\{n_{k_m}\}$ s.t. $f_{n_{k_m}} \to f$ μ-a.e. Then, since $\{g_n\}$ converges to g in measure, so does $\{g_{n_{k_m}}\}$. Therefore, there is a subsequence $\{n_{k_{m_j}}\}$ so that $g_{n_{k_{m_j}}} \to g$ μ-a.e. Since $f_{n_{k_m}} \to f$ μ-a.e., the same is true for the subsequence $\{f_{n_{k_{m_j}}}\}$.

Now, since $\{f_{n_{k_{m_j}}}\}$ and $\{g_{n_{k_{m_j}}}\}$ converge to f and g μ-a.e. (respectively), clearly $f_{n_{k_{m_j}}} g_{n_{k_{m_j}}} \to fg$ μ-a.e. (everywhere except on the union of the two null sets where convergence does not happen for the individual seuquences.) Since $\mu(X) < \infty$, this implies that $f_{n_{k_{m_j}}} g_{n_{k_{m_j}}} \to fg$ in measure as well.

But now, we've shown that for any subsequence $\{f_{n_k} g_{n_k}\}$ of $f_n g_n$, there exists a further subsequence $\{f_{n_{k_{m_j}}} g_{n_{k_{m_j}}}\}$ which converges to fg in measure. Since convergence in measure is described by the metric ρ (see problem 32), this implies that $f_n g_n \to fg$ in measure!

If that last sentence wasn't convincing, here's a formal justification: assume for a contradiction that $f_n g_n \not\to fg$. Then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k} g_{n_k}\}$ s.t. $\rho(f_{n_k} g_{n_k}, fg) \geq \epsilon$ for all k. But by the above, there's a subsequence of n_k along which $\rho(f_{n_k} g_{n_k}, fg)$ approaches 0, contradiction!