Selected solutions for MATH 4280 Assignment 6

49(a). Choose any $E \in \mathcal{M} \otimes \mathcal{N}$ with $(\mu \times \nu)(E) = 0$. Then by definition of product measure, there exist collections $A_n \in \mathcal{M}$, $B_n \in \mathcal{N}$ s.t. $E \subseteq \bigcup (A_n \times B_n)$ and $\sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) < \infty$.

This means that for each n, either $\mu(A_n)$ and $\nu(B_n)$ are both finite, $\mu(A_n) = \infty$ and $\nu(B_n) = 0$, or $\mu(A_n) = 0$ and $\nu(B_n) = \infty$. Define C, D, and E to be the sets of n for which these three cases occur, respectively. Define $A = \bigcup_{n \in C} A_n$, $B = \bigcup_{n \in C} B_n$, $Z = \bigcup_{n \in E} A_n$, and $Z' \in \bigcup_{n \in D} B_n$. Then by countable additivity, $\mu(Z) = \nu(Z') = 0$. By definition, A is σ-finite for μ and B is σ-finite for ν. And, by definition of A_n and B_n, $E \subseteq (A \times B) \cup (Z \times Y) \cup (X \times Z')$. Define $E_1 = E \cap (A \times B)$, $E_2 = E \cap (Z \times Y)$, and $E_3 = E \cap (X \times Z')$; clearly $E = E_1 \cup E_2 \cup E_3$. Since $(\mu \times \nu)(E) = 0$, by monotonicity $(\mu \times \nu)(E_1) = 0$.

We will show only that $\nu(E_x) = 0$ for μ-a.e. x, since the other proof is extremely similar. It will suffice to show this statement for E_1, E_2, and E_3, since $E_x = (E_1)_x \cup (E_2)_x \cup (E_3)_x$, and so if each of $\nu((E_1)_x)$, $\nu((E_2)_x)$, and $\nu((E_3)_x)$ are μ-a.e. equal to 0, the same will be true of $\nu(E_x)$ by finite subadditivity.

For E_1, we note that $E_1 \subseteq A \times B$, and so is contained in the product of two σ-finite measure spaces, namely the restrictions of μ and ν to A and B respectively. Therefore, by Theorem 2.36, $\nu((E_1)_x)$ is a measurable function on A, meaning that $\nu((E_1)_x)$ is also a measurable function on X (since it takes value 0 on A^c.) Also by Theorem 2.36,

$$0 = (\mu \times \nu)(E_1 \cap (A \times B)) = \int_A \nu((E_1)_x)\, d\mu = \int_X \nu((E_1)_x)\, d\mu.$$

(Recall that $\nu((E_1)_x) = 0$ on A^c.) Then, since $\nu((E_1)_x)$ is a nonnegative measurable function, by Proposition 2.16, it equals 0 μ-a.e.

For E_2, note that $(E_2)_x$ is only nonempty for $x \in Z$, and so $\nu(E_2)_x = 0$ for $x \in Z^c$, meaning that it is 0 μ-a.e. since $\mu(Z) = 0$.

For E_3, note that $(E_3)_x$ can only take values $\varnothing$ or Z', and in each case $\nu((E_3)_x) = 0$. So, in fact $\nu((E_3)_x) = 0$ everywhere.

This completes the proof, as described above.

Extra problem: First we rewrite using Lebesgue integral:

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right)\, dxdy = \int_T x^{-3/2} \cos\left(\frac{\pi y}{2x}\right)\, d(m^2) = \int x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) \chi_T\, d(m^2),$$

where $T = \{(x, y)\ :\ 0 \leq y \leq 1, y \leq x \leq 1\}$. We first apply Tonelli to show that $f(x, y) := x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) \chi_T$ is in $L^1(m^2)$:

$$\begin{aligned}\int |f|\, d(m^2) &\leq \int x^{-3/2} \chi_T\, d(m^2) = \int_0^1 \int_y^1 x^{-3/2}\, dxdy \\ &= \int_0^1 \left(-2x^{-1/2}\Big|_y^1\right) dy = \int_0^1 (-2 + 2y^{-1/2})\, dy = -2y + 4y^{1/2}\big|_0^1 = 2 < \infty.\end{aligned}$$

Now we can apply Fubini to the original integral:

$$\int f\, d(m^2) = \int \left(\int f_x \; dm(y) \right) dm(x) = \int \left(\int x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) (\chi_T)_x \; dm(y) \right) dm(x)$$
$$= \int \left(\int x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) \chi_{[0,x]} \; dm(y) \right) \chi_{[0,1]} dm(x) = \int \left(\int_0^x x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dy \right) \chi_{[0,1]} dm(x)$$
$$= \int \left(\frac{2}{\pi} x^{-1/2} \sin\left(\frac{\pi y}{2x}\right) \Big|_0^x \right) \chi_{[0,1]} dm(x) = \int \frac{2}{\pi} x^{-1/2} \chi_{[0,1]} \; dm(x) = \int_0^1 \frac{2}{\pi} x^{-1/2} \; dx$$
$$= \frac{4}{\pi} x^{1/2} \Big|_0^1 = \frac{4}{\pi}.$$