Selected solutions for MATH 4280 Assignment 7

2.3.12. Suppose that $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$, and that all measures are σ-finite. We first must show that $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$. Consider any set $C \in \mathcal{M}_1 \otimes \mathcal{M}_2$ with $(\mu_1 \times \mu_2)(C) = 0$. Then, by Proposition 2.36,

$$0 = (\mu_1 \times \mu_2)(C) = \int \mu_2(C_x)\ d\mu_1(x).$$

But $\mu_2(C_x)$ is a nonnegative measurable function in x, and so its integral equaling 0 forces it to be 0 μ_1-a.e. Therefore, there is a μ_1-null set A so that $\mu_2(C_x) = 0$ for all $x \in A$. Since $\nu_1 << \mu_1$, A is also a ν_1-null set, and since $\nu_2 << \mu_2$, $\nu_2(C_x) = 0$ for all $x \in A$. Therefore,

$$(\nu_1 \times \nu_2)(C) = \int \nu_2(C_x)\ d\nu_1(x) = 0,$$

since the function being integrated is 0 ν_1-a.e. We have then shown that $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$.

Since $\nu_1 << \mu_1$, $\nu_2 << \mu_2$, and $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$, each pair has a Radon-Nikodym derivative, which we denote by $f \in L^+(\mu_1)$, $g \in L^+(\mu_2)$, and $h \in L^+(\mu_1 \times \mu_2$ for brevity. Each is unique by the Lebesgue-Radon-Nikodym Theorem, and so to show that $h(x,y) = f(x)g(y)$, we must only show that fg satisfies the definition of a Radon-Nikodym derivative. In other words, we must show that for any $C \in \mathcal{M}_1 \otimes \mathcal{M}_2$, $(\nu_1 \times \nu_2)(C) = \int f(x)g(y)\chi_C\ d(\mu_1 \times \mu_2)$. (Call this identity (*).) We first prove (*) for rectangles:

$$(\nu_1 \times \nu_2)(A \times B) = \nu_1(A)\nu_2(B) = \left(\int f\chi_A\ d\mu_1\right)\left(\int g\chi_B\ d\mu_2\right)$$
$$= \int f(x)\chi_A(x)g(y)\chi_B(y)\ d(\mu_1 \times \mu_2) = \int h(x,y)\chi_{A\times B}(x,y)\ d(\mu_1 \times \mu_2).$$

Then, clearly (*) follows for finite disjoint unions of rectangles by additivity of the integral. But we have then shown that the measure ρ defined by $\rho(C) = \int f(x)g(y)\chi_C\ d(\mu_1 \times \mu_2)$ agrees with $\nu_1 \times \nu_2$ on all sets in the algebra of finite disjoint unions of rectangles, and so since $\nu_1 \times \nu_2$ is σ-finite, by Caratheodory ρ and $\nu_1 \times \nu_2$ agree for all sets. But this means that (*) holds for all $C \in \mathcal{M}_1 \otimes \mathcal{M}_2$, proving that $h(x,y) = f(x)g(y)$ and completing the proof.

2.3.13(a). $m << \mu$ because the only null sets for μ is the empty set, and $m(\varnothing) = 0$. Assume for a contradiction that there exists $f \geq 0$ so that $dm = f\ d\mu$. Then in particular, $1 = m([0,1]) = \int f\ d\mu$. However, for all n,

$$\int f\ d\mu \geq \int \frac{1}{n}\chi_{\{x\ :\ f(x) > \frac{1}{n}\}} = \frac{1}{n}\mu(\{x\ :\ f(x) > \frac{1}{n}\}) = \frac{1}{n}|\{x\ :\ f(x) > \frac{1}{n}\}|.$$

Therefore, for all n, $|\{x : f(x) > \frac{1}{n}\}| \leq n$, implying that $\{x : f(x) > 0\} = \bigcup\{x : f(x) > \frac{1}{n}\}$ is countable. Denote this set by C. Since $dm = f\ d\mu$,

$$m([0,1] \setminus C) = \int f\chi_{[0,1]\setminus C}\ d\mu = \int 0\ d\mu = 0.$$

This is however, a, contradiction, since $m(C) = m([0,1]) - m(C) = 1 - 0 = 1$ since C is countable. Therefore, we cannot represent $dm = f\ d\mu$.

2.3.13(b). Assume for a contradiction that μ has such a decomposition; then $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho << m$. By definition of mutual singularity, there exists a Lebesgue null set N so that $\lambda([0,1] \setminus N) = 0$. Since $m([0,1]) = 1$, $m([0,1] \setminus N) = 1 > 0$, and so there exists $x \in [0,1] \setminus N$. Then, $\lambda(\{x\}) = 0$ (since $x \in [0,1] \setminus N$ and $\lambda([0,1] \setminus N) = 0$) and $\rho(\{x\}) = 0$ (since $m(\{x\}) = 0$ and $\rho << m$. This implies that $\mu(\{x\}) = 0$, a contradiction to the definition of counting measure. Therefore, μ has no Lebesgue decomposition with respect to m.