MATH 4280 Practice Final Exam

Name: ____

Instructions: You may not use any instructional aids (book, notes, etc.) on this exam. You may use any fact proven in class or the textbook without proof unless an entire problem consists of proving such a fact. If you have ANY question about whether you need to justify a fact, or if you think a problem is unclear or incorrect, please ask me!

1. Prove that if $f \in L^+(X, \mathcal{M}, \mu)$ and $\int f \, d\mu = 0$, then $f = 0$ μ-a.e.

2. (a) Prove that if $f_n \to f$ μ-a.e. and $\mu(X) < \infty$, then $f_n \to f$ in measure.

(b) Show that if $\mu(X) = \infty$, the above implication may not hold.

3. For any $f \in L^+(X, \mathcal{M}, \mu)$, define a measure ν by $\nu(E) = \int_E f \, d\mu$ for all $E \in \mathcal{M}$. (You DO NOT have to prove that this is a measure!). Prove that for any $g \in L^+(X, \mathcal{M}, \mu)$, $\int g \, d\nu = \int fg \, d\mu$.

4. If $(X, \mathcal{M}, \mu)$ is a measure space and μ is complete, show that the following statement is true: for any functions f, g where f is measurable and $\mu(\{x : f(x) \neq g(x)\}) = 0$, g is also measurable. (This statement is in fact equivalent to completeness, so you need to use completeness in the proof somewhere!)

5. Suppose that μ, ν are σ-finite measures where $\mu << \nu$ and $\nu << \mu$. Prove that for almost every x (with respect to either μ or ν; the statements are equivalent),

$$\frac{d\mu}{d\nu} \cdot \frac{d\nu}{d\mu} = 1,$$

where $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$ are Radon-Nikodym derivatives.

6. Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and that f_n approaches 0 in the L^1 metric with respect to $\mu \times \nu$. Show that there exists a subsequence f_{n_k} with the property that for μ-a.e. x, the cross-section functions $(f_n)_x$ approach 0 in the L^1 metric with respect to ν.