MATH 4280 Practice Midterm Exam

Name: ____

Instructions: You may not use any instructional aids (book, notes, etc.) on this exam. You may use any fact proven in class or the textbook without proof unless an entire problem consists of proving such a fact. If you have ANY question about whether you need to justify a fact, or if you think a problem is unclear or incorrect, please ask me!

1. (a) Prove that for a set X, $\mathcal{E}$ an arbitrary subset of $\mathcal{P}(X)$, and $\mathcal{M}$ a σ-algebra on X, if $\mathcal{E} \subseteq \mathcal{M}$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}$. (Recall that $\mathcal{M}(\mathcal{E})$ is the σ-algebra generated by $\mathcal{E}$.)

(b) Use (a) to show that if $\mathcal{C}$ is the collection of bounded closed intervals in $\mathbb{R}$ and $\mathcal{O}$ is the collection of bounded open intervals in $\mathbb{R}$, then $\mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{O})$.

2. Prove that for a set X and σ-algebra $\mathcal{M}$ on X, if $\mu : \mathcal{M} \to [0, \infty]$ is finitely additive and continuous from below, then it is countably additive. (Recall that μ is continuous from below if for any sequence of sets $A_1 \subseteq A_2 \subseteq A_3 \ldots$, it is the case that $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.)

3. If X is a set, $\mathcal{M}$ is a σ-algebra on X, and $f, g : X \to \mathbb{R}$ are measurable functions (where $\mathbb{R}$ is endowed with the Borel σ-algebra), prove that $\max(f, g) : X \to \mathbb{R}$ is also a measurable function.

4. If μ is an arbitrary measure on $\mathbb{R}$ with associated σ-algebra $\mathcal{M}$, $f : \mathbb{R} \to [0, \infty]$ is in $L^+(\mathbb{R}, \mathcal{M}, \mu)$, and $\int f \, d\mu < \infty$, prove that for any $\epsilon > 0$ there exists an N for which $\int_{[-N,N]^c} f \, d\mu < \epsilon$.

5. If X is a set, $\mathcal{E}$ is an arbitrary subset of $\mathcal{P}(X)$ containing $\varnothing$ and X, and ρ is a function from $\mathcal{E}$ to $[0, \infty]$ with $\rho(\varnothing) = 0$, prove that the function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ defined by

$$\mu^*(A) = \inf \left\{ \sum_{n\in\mathbb{N}} \rho(E_n) \ : \ A \subseteq \bigcup_{n\in\mathbb{N}} E_n, \ \forall n \ E_n \in \mathcal{E} \right\}$$

is countably subadditive.

6. If $(X, \mathcal{M}, \mu)$ is a finite measure space (i.e. $\mu(X) < \infty$) and $f_n : X \to \mathbb{R}$ is a sequence of measurable functions (where $\mathbb{R}$ is endowed with the Borel σ-algebra) which approach a limit function f μ-a.e., prove that

$$\lim_{n\to\infty} \mu(\{x \ : \ |f_n(x) - f(x)| > 1\}) = 0.$$

(HINT: I told you that $\mu(X) < \infty$ for the purposes of applying continuity from above.))