MATH 4280 Practice Midterm Exam

Name: ______________________________

Instructions: You may not use any instructional aids (book, notes, etc.) on this exam. You may use any fact proven in class or the textbook without proof unless an entire problem consists of proving such a fact. If you have ANY question about whether you need to justify a fact, or if you think a problem is unclear or incorrect, please ask me!

3. If X is a set, $\mathcal{M}$ is a σ-algebra on X, and $f, g : X \to \mathbb{R}$ are measurable functions (where $\mathbb{R}$ is endowed with the Borel σ-algebra), prove that $\max(f, g) : X \to \mathbb{R}$ is also a measurable function.

Solution: $\max(f, g) = f\chi_{M_f} + g\chi_{M_g}$, where

$$M_f = \{x \ : \ f(x) \geq g(x)\} \text{ and } M_g = \{x \ : \ g(x) > f(x)\}.$$

Clearly $M_g = M_f^c$, and $M_f = (f - g)^{-1}([0, \infty))$. Since f and g are measurable, so is $f - g$, and $[0, \infty) \in \mathcal{B}(\mathbb{R})$, so $M_f \in \mathcal{M}$. By closure under complements, $M_g \in \mathcal{M}$ as well. Then, χ_{M_f} and χ_{M_g} are both measurable functions, implying that $\max(f, g)$ is as well since it is a sum of products of measurable functions.

4. If μ is an arbitrary measure on $\mathbb{R}$ with associated σ-algebra $\mathcal{M}$, $f : \mathbb{R} \to [0, \infty]$ is in $L^+(\mathbb{R}, \mathcal{M}, \mu)$, and $\int f \ d\mu < \infty$, prove that for any $\epsilon > 0$ there exists an N for which $\displaystyle\int_{[-N,N]^c} f \ d\mu < \epsilon$.

Solution: Consider the sequence of functions $g_n := f\chi_{[-n,n]}$. Then $g_n \to f$ pointwise, $|g_n| \leq f$, and $g_1 \leq g_2 \leq g_3 \ldots$. This means that you can apply either the Monotone Convergence Theorem or the Dominated Convergence Theorem to see that

$$\int g_n \ d\mu = \int_{[-n,n]} f \ d\mu \to \int f \ d\mu.$$

But then by additivity of integrals,

$$\int_{[-n,n]^c} f \ d\mu = \int f\chi_{[-n,n]^c} \ d\mu = \int (f - f\chi_{[-n,n]}) \ d\mu = \int (f - g_n) \ d\mu = \int f \ d\mu - \int g_n \ d\mu \to 0.$$

Now the problem is complete by definition of convergence.

6. If $(X, \mathcal{M}, \mu)$ is a finite measure space (i.e. $\mu(X) < \infty$) and $f_n : X \to \mathbb{R}$ is a sequence of measurable functions (where $\mathbb{R}$ is endowed with the Borel σ-algebra) which approach a limit function f μ-a.e., prove that

$$\lim_{n\to\infty} \mu(\{x \ : \ |f_n(x) - f(x)| > 1\}) = 0.$$

(HINT: I told you that $\mu(X) < \infty$ for the purposes of applying continuity from above.))

Solution: Here is a solution that requires no theorems about convergence of integrals. For each n, define

$$S_n = \{x \ : \ |f_n(x) - f(x)| > 1\}.$$

We could write $S_n = (f_n - f)^{-1}([-1, 1]^c)$, so clearly all S_n are measurable since f and all f_n are. Now, define

$$U_n = \bigcup_{k \geq n} S_k.$$

By definition of μ-a.e. convergence, $\bigcap_{n=1}^{\infty} U_n$ is a null set. Also, clearly $U_1 \supseteq U_2 \supseteq \ldots$. Therefore, by continuity from above, (since $\mu(X) < \infty$), $\mu(U_n) \to 0$. Since $S_n \subset U_n$, by monotonicity $0 \leq \mu(S_n) \leq \mu(U_n)$, and so $\mu(S_n) \to 0$ as well, completing the proof.