MATH 4290 Homework Assignment 1 Solutions

• If (X, T) is a non-invertible dynamical system and $x \in X$ is recurrent, prove that Tx is recurrent.

Solution: Assume that (X, T) is a non-invertible topological dynamical system, and $x \in X$ is recurrent, i.e. that for any open $U \ni x$, there exists $n \in \mathbb{N}$ s.t. $T^n x \in U$. Now, consider Tx and any open set U containing Tx. Then $V := T^{-1}U$ is open by continuity of T, and contains x. By recurrence, there then exists n so that $T^n x \in V$. But then $T^n(Tx) = T(T^n x) \in TV \subseteq U$, and since U was arbitrary, Tx is recurrent.

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• If (X, T) is a non-invertible dynamical system, $x \in X$ is recurrent, and U is an open set containing x, prove that $R_U(x) := \{n \in \mathbb{N} \ : \ T^n x \in U\}$ is infinite.

Solution: Choose any such (X, T), recurrent $x \in X$, and $U \ni x$. Problem 1 shows via a simple induction argument that $T^n x$ is recurrent for all $n \in \mathbb{N}$. Assume for a contradiction that the set $\{n \in \mathbb{N} \ : \ T^n x \in U\}$ is finite. Then it has a greatest element, call it N. Then $T^N x$ is recurrent, and $T^N x \in U$, and so by definition there exists $k \in \mathbb{N}$ so that $T^k(T^N x) \in U$. But then $T^{k+N} x \in U$, contradicting the maximality of N.

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• If (X, T) is a minimal dynamical system and (Y, S) is conjugate to (X, T), prove that (Y, S) is also minimal.

Solution: Either definition of minimality will work for this problem. Probably the easiest solution is to use the "no proper subsystems definition" and prove the contrapositive, i.e. we'll prove that if (Y, S) is not minimal, then (X, T) is not minimal. Assume that (Y, S) is not minimal; then there is a nonempty compact S-invariant proper subset of Y, call it $\varnothing \neq Y' \subsetneq Y$. Denote by ϕ the conjugacy from X to Y. Then, we claim that $\phi^{-1}(Y')$ is a nonempty proper compact T-invariant proper subset of X, which implies that (X, T) is not minimal.

The fact that $\varnothing \neq \phi^{-1}(Y') \subsetneq X$ is obvious since ϕ is a bijection and $\varnothing \neq Y' \subsetneq Y$. To prove that $\phi^{-1}(Y')$ is T-invariant, consider $x \in \phi^{-1}(Y')$. Then $\phi(x) \in Y'$, and since Y' is S-invariant, $S\phi(x) \in Y'$. By the definition of conjugacy, $S\phi(x) = \phi(Tx)$, so $\phi(Tx) \in Y'$, implying that $Tx \in \phi^{-1}(Y')$. So, $\phi^{-1}(Y')$ is T-invariant, completing the proof.

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• Prove that there exists $n \in \mathbb{N}$ so that the decimal expansion of 2^n begins with 777, and give a provable upper bound on n.

Solution: An integer n starts with 777 if and only if there exists $k \in \mathbb{N}$ so that $n \in [7.77 \cdot 10^k, 7.78 \cdot 10^k)$, which is true if and only if $\log_{10} n \in [k + \log_{10} 7.77, k + \log_{10} 7.78)$. Therefore, 2^m starts with 777 if and only if $\log_{10}(2^m) \in [k + \log_{10} 7.77, k + \log_{10} 7.78)$ for some $k \in \mathbb{N}$, or equivalently, if $m \log_{10} 2 \pmod 1 \in [\log_{10} 7.77, \log_{10} 7.78)$. But $m \log_{10} 2$ can be written as $(T_{\log_{10} 2})^m 0$, where $(\mathbb{T}, T_{\log_{10} 2})$ is the rotation by $\log_{10} 2$ on $\mathbb{T}$ as defined in class.

We claim that $\log_{10} 2 \notin \mathbb{Q}$. Indeed, if $\log_{10} 2 = \frac{p}{q}$, then $2^{p/q} = 10$, so $2^q = 10^p$. This is impossible, since 5 divides the right-hand side and not the left-hand side. Since irrational circle rotations are minimal, all orbits under $T_{\log_{10} 2}$ are dense. Therefore, there is an $m \in \mathbb{N}$ such that $(T_{\log_{10} 2})^m 0 \in [\log_{10} 7.77, \log_{10} 7.78)$, and so 2^m starts with 777 as argued above.

Getting an explicit upper bound on m is a little harder. We essentially need to reconstruct the reasoning from the proof of denseness of orbits. In other words, we start by taking ϵ the length of the interval $[\log_{10} 7.77, \log_{10} 7.78)$, which is about $0.00056 > \frac{1}{2000}$. This means that if we break the unit circle into 2000 intervals of length $\frac{1}{2000}$, two elements of $\{0, \log_{10} 2 \pmod 1, 2\log_{10} 2 \pmod 1, \ldots, 2000 \log_{10} 2 \pmod 1\}$ are in the same interval, and taking their difference yields $0 \le i \le 2000$ so that $i \log_{10} 2 \pmod 1$ is within distance $\frac{1}{2000} < \epsilon$ of 0. A quick computer search finds an explicit such i; $485 \log_{10} 2 \pmod 1 = 0.99954\ldots$. Multiples of $485 \log_{10} 2$ will then be ϵ-dense throughout the unit circle, and so one will lie in $[\log_{10} 7.77, \log_{10} 7.78)$. Since $485 \log_{10} 2 \pmod 1$ is slightly less than 1, these multiples need to move a distance of $1 - \log_{10} 7.78$ to get to the interval, and so there is definitely a number of the form $485 j \log_{10} 2 \pmod 1$, $j \le \frac{1 - \log_{10} 7.78}{1 - 485 \log_{10} 2 \pmod 1}$ in the interval in question. We can then choose the upper bound $485 \cdot \frac{1 - \log_{10} 7.78}{1 - 485 \log_{10} 2 \pmod 1}$, which is approximately $116953.2\ldots$.

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• Define (X, T) by $X = \mathbb{T}^2$, which you can think of as $[0,1)^2$ with the top/bottom edges identified and left/right edges identified, and $T : (x, y) \mapsto (x + \alpha, y + \alpha)$ for $\alpha \notin \mathbb{Q}$. Describe, with proof, all minimal subsystems of (X, T).

Solution: For readability, we omit $\pmod 1$ in all future calculations. For every $c \in [0, 1)$, define the set $D_c := \{(x, x + c) \ : \ x \in [0, 1)\}$. We claim that for every c, (D_c, T) is a minimal subsystem, and that these are all minimal subsystems. First, we verify that each D_c is closed and T-invariant. T-invariance is easy; clearly if $(x, x + c) \in D_c$, then $T(x, x + c) = (x + \alpha, x + \alpha + c) \in D_c$ as well. For closed, take a sequence $(x_n, x_n + c)$ in D_c converging to a limit (y, z). Clearly $x_n \to y$ and $x_n + c \to z$, so by algebraic laws of limits, $z = y + c$ and the limit $(y, z) \in D_c$.

To see that each (D_c, T) is minimal, we describe a conjugacy $(D_c, T) \to (\mathbb{T}, T_\alpha)$, where T_α is the usual circle rotation by α. The conjugacy is simple: for any $(x, x + c) \in D_c$, we define $\phi(x, x + c) = x$. This is a projection map (to the first coordinate), and therefore obviously continuous. The inverse map is ϕ^{-1} defined by $\phi^{-1}(x) = (x, x + c)$, which is clearly continuous; if $x_n \to x$, then

$\phi^{-1}(x_n) = (x_n, x_n+c) \to (x, x+c) = \phi^{-1}(x)$. It remains to check commutativity of the diagram:

$$\phi(T(x, x+c)) = \phi(x+\alpha, x+c+\alpha) = x+\alpha = T_\alpha(x) = T_\alpha(\phi(x, x+c)).$$

So, ϕ is indeed a conjugacy, meaning that all (D_c, T) are minimal since $(\mathbb{T}, T_\alpha)$ is known to be minimal from class.

All that remains is to see that there are no other minimal subsystems, but this is easy; any other minimal subsystem would need to be disjoint from all D_c from results from class, and $\bigcup_c D_c = \mathbb{T}^2 = X$. So, there can be no other minimal subsystem within X.

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• If (X, T) is a dynamical system and (M, T) is a nonempty subsystem, prove that (M, T) is the unique minimal subsystem of (X, T) if and only if every subsystem of (X, T) contains (M, T).

Solution: $\Longrightarrow$: Assume that (M, T) is the unique minimal subsystem of (X, T). Consider any subsystem (Z, T) of (X, T). Then (Z, T) is a dynamical system, and so contains a minimal subsystem (N, T). Then (N, T) is a minimal subsystem of (X, T), and by assumption $N = M$. Therefore, (Z, T) contains (M, T).

$\Longleftarrow$: Assume that every subsystem of (X, T) contains (M, T), and assume for a contradiction that (M, T) is not the unique minimal subsystem of (X, T). Since (X, T) contains at least one minimal subsystem from results in class, there must be another minimal subsystem (M', T) within (X, T). However, we showed that minimal subsystems are disjoint, and so (M', T) does not contain (M, T), a contradiction to our assumption. Our assumption was then wrong, and (M, T) is the unique minimal subsystem of (X, T).

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