MATH 4290 Homework Assignment 2 Solutions

• We can define the two-sided full shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$ in almost the same way as the (one-sided) full shift $(\{0,1\}^{\mathbb{N}}, \sigma)$ from class. (It's convenient to view $\{0,1\}^{\mathbb{Z}}$ as a metric space with the metric

$$d((x_n),(y_n)) = 2^{-\max\{i\in\mathbb{N} \ : \ x_j=y_j \text{ for all } j \text{ satisfying } |j|<|i|\}}.)$$

Prove that σ is a homeomorphism on $\{0,1\}^{\mathbb{Z}}$, that the two-sided full shift factors onto the one-sided full shift, and that the two-sided full shift is NOT conjugate to the one-sided full shift.

Solution: It is trivial that σ is a bijection on the two-sided full shift. By definition, $d(\sigma(x_n),\sigma(y_n)), d(\sigma^{-1}(x_n),\sigma^{-1}(y_n)) \leq 2d(x_n,y_n)$ (this is because $\max\{i \in \mathbb{N} \ : \ x_j = y_j \text{ for all } j \text{ satisfying } |j| < |i|\}$ cannot decrease by more than 1 by a single shift of both sequences in either direction), which immediately shows that σ is a homeomorphism.

To show that the two-sided full shift factors onto the full one-sided full shift, simply define the restriction ϕ by $\phi((x_n)_{n\in\mathbb{Z}}) = (x_n)_{n\in\mathbb{N}}$. To see that ϕ is continuous, just note that by definition, $d(\phi(x_n),\phi(y_n)) \leq d((x_n),(y_n))$ (this is because two sequences agreeing on n units to either side of the origin trivially also agree on the first n units after the origin.) Commuting of the diagram is nearly as simple: both $\phi(\sigma((x_n)_{n\in\mathbb{Z}}))$ and $\sigma(\phi((x_n)_{n\in\mathbb{Z}}))$ are equal to $x_2x_3x_4\ldots$.

Finally, the two-sided full shift is invertible, and the one-sided full shift is not, so these two systems cannot be conjugate. Formally, since the one-sided full shift is not invertible, there exist $x \neq y$ so that $\sigma(x) = \sigma(y)$ in the one-sided full shift. If there were a conjugacy ψ from the one-sided full shift to the two-sided full shift, then $\psi(x) \neq \psi(y)$, but $\sigma(\psi(x)) = \psi(\sigma(x)) = \psi(\sigma(y)) = \sigma(\psi(y))$, contradicting invertibility of the two-sided full shift.

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• A dynamical system (X,T) is called **topologically transitive** if it has a transitive point, i.e. a point with an orbit which is dense in X. Prove that the following definition is equivalent: (X,T) is topologically transitive if for all nonempty open sets A,B, there exists n so that $A \cap T^nB \neq \varnothing$.

Solution: The direction "new definition implies dense orbit" is in your textbook as Proposition 2.2.1, with a proof provided there. For the other direction, assume that there exists $x \in X$ with a dense orbit. Consider any nonempty open sets A,B. By definition of dense orbit, there exist m,n s.t. $T^mx \in A$ and $T^nx \in B$. But then $T^mx \in A \cap T^{m-n}B$, so $A \cap T^{m-n}B \neq \varnothing$. (This proof requires T to be a homeomorphism; in fact if T is non-invertible, then you must assume that x not only has a dense orbit, but a dense forward orbit, i.e. $\mathcal{O}^+(x) := \{T^nx \ : \ n \in \mathbb{N}\}$ must be dense.)

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• Show that the map $\phi : \{0,1\}^{\mathbb{N}} \to [0,1)$ defined by $\phi(x_1 x_2 \ldots) = \sum_{n=1}^{\infty} x_i 2^{-i}$ is continuous, with the product topology on $\{0,1\}^{\mathbb{N}}$ and the usual (Borel) topology on $[0,1)$.

Solution: Suppose that sequences $x^{(k)}$, where $x^{(k)} = x_1^{(k)} x_2^{(k)} \ldots$, converge to a limit sequence $x = x_1 x_2 \ldots$ in the product topology on $\{0,1\}^{\mathbb{N}}$. As discussed in class, this means that $x^{(k)}$ agrees with x on longer and longer initial segments as $k \to \infty$. Then, for any $\epsilon > 0$, choose K so that for $k > K$, $x_i^{(k)} = x_i$ for $1 \le i \le \lceil -\log_2 \epsilon \rceil$. This means that for any $k > K$,

$$|\phi(x^{(k)}) - \phi(x)| = \left| \sum_{i=1}^{\infty} 2^{-i}(x_i^{(k)} - x_i) \right| \le \sum_{i=1}^{\infty} 2^{-i} |x_i^{(k)} - x_i|.$$

Note that the first $\lceil -\log_2 \epsilon \rceil$ terms of this are 0, and so the entire sum is bounded from above by

$$\sum_{i=\lceil -\log_2 \epsilon \rceil + 1}^{\infty} 2^{-i} < \epsilon.$$

Therefore, $\phi(x^{(k)})$ converges to $\phi(x)$, proving continuity of ϕ.

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• For the full shift $(\{0,1\}^{\mathbb{N}}, \sigma)$, prove that the set of points with dense orbits is uncountable. (**Optional challenge question:** show that this set is residual, i.e. that its complement is a countable union of nowhere dense sets.)

Solution: As we discussed in class, any $x \in \mathbb{T}$ with a binary expansion which contains every finite string of 0s and 1s somewhere has a dense orbit under σ. There are many ways to show that this set is uncountable. One is to deal with permutations of the set of finite strings, but this is slightly tricky. An easier solution is to enumerate the set of finite strings as $(w_n)_{n \in \mathbb{N}}$, and define an uncountable set S of points by $.w_1 * w_2 * w_3 * \ldots$, where each of the $*$ symbols can be filled with either 0 or 1. Each of these points will clearly have a dense orbit. Also, there is a clear injection from $\{0,1\}^{\mathbb{N}}$ to S, by simply placing any $0-1$ sequence in the starred locations. Therefore, S is uncountable.

Solution to challenge problem: We claim that for any finite string w of 0s and 1s, the set $A_w := \{x \ : \ x \text{ does not contain } w\}$ is nowhere dense in the full shift. If we show this, we will be finished with the problem, as by our discussions in class, x has dense orbit if and only if contains every finite string w, which is true if and only if $x \in (\bigcup_w A_w)^c$.

Choose any finite string w, and any open set U. Since cylinder sets form a basis for the topology on $\{0,1\}^{\mathbb{N}}$, U contains some cylinder set $[v]$. The cylinder set $[v]$ clearly contains the cylinder set $[vw]$, where vw is just the concatenation of v and w, i.e. the word obtained by placing w immediately to the right of v. However, the cylinder set $[vw]$ consists only of sequences containing vw, each of which trivially contains w, and so $[vw] \subset A_w^c$. So, U contained the open set $[vw]$ disjoint from A_w. Since U was arbitrary, this shows that A_w is nowhere dense, and since w was arbitrary, we are done.

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• Show that if $\frac{\alpha}{\beta} \notin \mathbb{Q}$, there does not exist a factor map from $(\mathbb{T}, T_\alpha)$ to $(\mathbb{T}, T_\beta)$. (Hint: you may use the fact that $T_{\alpha,\beta}$ is minimal for this problem, even though we haven't finished a proof in class.)

Solution: Assume for a contradiction that ϕ is a factor map from $(\mathbb{T}, T_\alpha)$ to $(\mathbb{T}, T_\beta)$. There are multiple cases to consider. The first is if $\alpha \in \mathbb{Q}$, say $\alpha = \frac{p}{q}$. Then, clearly by assumption $\beta \notin \mathbb{Q}$. Every point $x \in \mathbb{T}$ satisfies $T_\alpha^q x = x$, and so by the commutativity of the diagram, $T_\beta^q(\phi(x)) = \phi(T_\alpha^q x) = \phi(x)$. However, since $\beta \notin \mathbb{Q}$, this is impossible, so we have a contradiction, and such ϕ could not exist.

The second case is if $\alpha \notin \mathbb{Q}$ and $\beta \in \mathbb{Q}$, say $\beta = \frac{p}{q}$. But then $(\mathbb{T}, T_\alpha)$ is minimal and $(\mathbb{T}, T_\beta)$ is not, and it's easily checked (it's the same proof as Problem 3 of Assignment 1) that a factor of a minimal system must be minimal. We therefore again have a contradiction to the existence of ϕ.

Finally, we consider the most interesting case, where $\alpha, \beta \notin \mathbb{Q}$. Then, as discussed in class, $T_{\alpha,\beta}$ is minimal. Therefore, $(0, 0.5)$ is in the orbit closure of $(0,0)$ under $T_{\alpha,\beta}$, i.e. there exists a sequence of integers (n_k) so that $T_{\alpha,\beta}^{n_k}(0,0) \to (0, 0.5)$. This means that $n_k\alpha = T_\alpha^{n_k} 0 \to 0 \pmod 1$ and $n_k\beta = T_\beta^{n_k} 0 \to 0.5 \pmod 1$. Now, consider any $x \in \mathbb{T}$. Since $n_k\alpha \to 0 \pmod 1$, $T_\alpha^{n_k} x \to x \pmod 1$. Since ϕ is continuous, we then see that

$$\phi(T_\alpha^{n_k} x) \to \phi(x).$$

However, since $n_k\beta \to 0.5 \pmod 1$, $T_\beta^{n_k} y \to y + 0.5 \pmod 1$ for all $y \in \mathbb{T}$. Combining this with commutativity of the diagram yields

$$\phi(T_\alpha^{n_k} x) = T_\beta^{n_k} \phi(x) \to \phi(x) + 0.5 \neq \phi(x) \pmod 1.$$

The sequence $(\phi(T_\alpha^{n_k} x))$ cannot approach two limits, so we have our final contradiction to the existence of ϕ.

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• If x is a uniformly recurrent point in a dynamical system (X, T), and X is a compact metric space, prove that $(\overline{\mathcal{O}(x)}, T)$ is a minimal system.

Solution: Assume that x is uniformly recurrent. Consider any nonempty open set U in $\overline{\mathcal{O}(x)}$; then there exists U' open in X so that $U' \cap \overline{\mathcal{O}(x)} = U$. Since U' is an open set intersecting the closure of $\mathcal{O}(x)$, it must contain a point in $\mathcal{O}(x)$, say $T^m x \in U'$; clearly then $T^m x \in U$ as well, and so x is contained in the set $T^{-m}U$, which is open by continuity of T. Since X is a compact metric space (in fact we'd only need that X is T_4), there exists an open set V containing x so that $\overline{V} \subset T^{-m}U$.

By uniform recurrence of x, there exists N so that for every n, at least one of $T^n x, \ldots, T^{n+N} x$ is in V. Now, consider an arbitrary point $y \in \overline{\mathcal{O}(x)}$. By definition, there exists a sequence (n_k) so that $T^{n_k} x \to y$. For each k, there exists $0 \leq i_k \leq N$ so that $T^{n_k + i_k} x \in V$. Since there are only finitely many choices for i_k and infinitely many k, there exists a further subsequence (n_{k_j}) so that for all j, i_{k_j} is the same number, let's call it i. Therefore, for all j, $T^{n_{k_j}+i} x \in V$. However, $T^{n_{k_j}+i} x = T^i T^{n_{k_j}} x$. We know that $T_{n_{k_j}} x \to y$ since it's a subsequence of $T^{n_k} x$, and T^i is continuous. Therefore, $T^{n_{k_j}+i} x \to T^i y$. Since each term in this sequence was in V, by definition of closure $T^i y \in \overline{V} \subset T^{-m}U$. Finally, this means that $T^{i+m} y \in U$, i.e. the orbit of y contains a point in U. Since U was arbitrary, the orbit of Y is dense in $\overline{\mathcal{O}(x)}$. Since $y \in \overline{\mathcal{O}(x)}$ was arbitrary, we've shown that all orbits in $(\overline{\mathcal{O}(x)}, T)$ are dense, and so $(\overline{\mathcal{O}(x)}, T)$ is minimal.

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• **Optional challenge question:** Find x in the full shift which is uniformly recurrent but not periodic. This will yield an infinite minimal subsystem $(\overline{\mathcal{O}(x)}, \sigma)$ within the full shift.

Solution: There are many solutions to this, but they're all quite tricky. (This really was a challenge problem!) Problem 4 on Assignment 3 outlines a specific way to get such x, and so the solution to that problem will serve as the easiest solution to this one.

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