# MATH 4290 Homework Assignment 3

Due on Thursday, October 4th, at the BEGINNING of class.

• Prove that if $(X, T)$ is topologically transitive and $(X, T)$ factors onto $(Y, S)$, then $(Y, S)$ is topologically transitive.

• Prove that if $(X, T)$ is expansive and $(Y, S)$ and $(X, T)$ are conjugate, then $(Y, S)$ is expansive.

• Prove that if $(X, T)$ and $(Y, S)$ are topologically mixing, then $(X \times Y, T \times S)$ is topologically mixing.

• At the end of class Thursday, we defined a way of symbolically coding orbits of a circle rotation $T_\alpha$ (for $\alpha \notin \mathbb{Q}$). For any $\alpha \notin \mathbb{Q}$, and any $x \in \mathbb{T}$, the "orbit coding sequence" $\psi_\alpha(x) \in \{0,1\}^{\mathbb{N}}$ is defined as follows.

For every $n \in \mathbb{N}$, define the $n$th bit $(\psi_\alpha(x))_n$ of $\psi_\alpha(x)$ to be 1 if $T_\alpha^n x \in [0, \alpha)$, and 0 otherwise, i.e. if $T_\alpha^n x \in [\alpha, 1)$. Then, each $x \in \mathbb{T}$ yields a sequence $\psi_\alpha(x)$. Prove that any such sequence $\psi_\alpha(x)$ is uniformly recurrent as a point of the full shift. Hint: what does it mean for $\psi_\alpha(x)$ to be in a cylinder set $[w]$, i.e. to start with a certain finite string of digits? Can you characterize the set $\{x \in \mathbb{T} \ : \ \psi_\alpha(x) \in [w]\}$?

• Suppose that $(X, T)$ is any invertible expansive topological dynamical system. I want you to construct a factor map from some two-sided symbolic system $(Y, \sigma)$ to $(X, T)$, where $Y \subseteq \{1, 2, \dots, N\}^{\mathbb{Z}}$ for some $N$. Here is an outline:

(a) Prove that for the expansiveness constant $\delta > 0$ of $(X, T)$, if $x, y$ have the property that $d(T^n x, T^n y) < \delta$ for all $n \in \mathbb{Z}$, then $x = y$.
(b) Use this to cover $X$ with a finite collection of closed balls $A_i$ so that, for any $x$, knowledge of a sequence $(k_n)_{n \in \mathbb{Z}}$ s.t. $T^n x \in A_{k_n}$ for all $n \in \mathbb{Z}$ uniquely determines $x$.
(c) Use this to construct a symbolic system $Y \subseteq \{1, 2, \dots, N\}^{\mathbb{Z}}$ and a factor map from $(Y, \sigma)$ to $(X, T)$. Remember that you must show that $Y$ is closed in the product topology and $\sigma$-invariant!