MATH 4290 Homework Assignment 3 Solutions

• Prove that if (X,T) is topologically transitive and (X,T) factors onto (Y,S), then (Y,S) is topologically transitive.

Solution: Suppose that (X,T) is topologically transitive and that ϕ is a factor map from (X,T) to (Y,S). Consider any nonempty open sets U,V in Y. By continuity and surjectivity of ϕ, $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty and open in X. Therefore, by transitivity of (X,T), there exists n so that $\phi^{-1}U \cap T^n\phi^{-1}V \neq \varnothing$. But then $\phi(\phi^{-1}U \cap T^n\phi^{-1}V) \neq \varnothing$, and by commutativity of the diagram,

$$\phi(\phi^{-1}U \cap T^n\phi^{-1}V) \subseteq U \cap \phi(T^n\phi^{-1}V) = U \cap \phi(\phi^{-1}S^nV) \subseteq U \cap S^nV,$$

and so $U \cap S^nV \neq \varnothing$, completing the proof.

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• Prove that if (X,T) is expansive and (Y,S) and (X,T) are conjugate, then (Y,S) is expansive.

Solution: Suppose that ϕ is a conjugacy from (X,T) to (Y,S). Then ϕ is a homeomorphism, and $\phi(Tx) = S(\phi(x))$ for all $x \in X$. Since (X,T) is expansive, there exists $\delta > 0$ so that for any $x \neq x'$ in X, $\exists n$ s.t. $d(T^nx, T^nx') > \delta$. Since ϕ^{-1} is continuous and Y is compact, it is uniformly continuous, and so there exists $\eta > 0$ so that for all $y, y' \in Y$, $d(y,y') < \eta \Longrightarrow d(\phi^{-1}x, \phi^{-1}y) < \delta$.

Now, suppose that $y \neq y'$ are points of X. Then, by injectivity of ϕ, $\phi^{-1}y \neq \phi^{-1}y'$ are points of X. Therefore, there exists n so that $d(T^n\phi^{-1}y, T^n\phi^{-1}y') > \delta$. Since ϕ is an isomorphism, $T^n\phi^{-1}y = \phi^{-1}(S^ny)$ and $T^n\phi^{-1}y' = \phi^{-1}(S^ny')$, so $d(\phi^{-1}(S^ny), \phi^{-1}(S^ny')) > \delta$. Then by definition of η, $d(S^ny, S^ny') \geq \eta$. Since $y \neq y'$ were arbitrary, this shows that η is a constant demonstrating that (Y,S) is expansive.

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• Prove that if (X,T) and (Y,S) are topologically mixing, then $(X \times Y, T \times S)$ is topologically mixing.

Solution: Suppose that (X,T) and (Y,S) are topologically mixing, and consider any open sets W, W' in $X \times Y$. By definition of the product topology, there exist open sets U, U' in X and V, V' in Y so that $W \supset U \times V$ and $W' \supset U' \times V'$. By definition of topological mixing of (X,T), there exists N so that for any $n > N$, $U \cap T^{-n}V \neq \varnothing$. By definition of topological mixing of (Y,S), there exists N' so that for any $n > N'$, $U' \cap S^{-n}V' \neq \varnothing$. Then, for any $n > \max(N,N')$, both $U \cap T^{-n}V$ and $U' \cap S^{-n}V'$ are nonempty, and so $(U \times U') \cap (T \times S)^{-n}(V \times V')$ is nonempty as well. But this implies that $W \cap (T \times S)W' \neq \varnothing$, and since W and W' were arbitrary, this implies that $(X \times Y, T \times S)$ is topologically mixing.

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• At the end of class Thursday, we defined a way of symbolically coding orbits of a circle rotation T_α (for $\alpha \notin \mathbb{Q}$). For any $\alpha \notin \mathbb{Q}$, and any $x \in \mathbb{T}$, the "orbit coding sequence" $\psi_\alpha(x) \in \{0,1\}^{\mathbb{N}}$ is defined as follows.

For every $n \in \mathbb{N}$, define the nth bit $(\psi_\alpha(x))_n$ of $\psi_\alpha(x)$ to be 1 if $T_\alpha^n x \in [0,\alpha)$, and 0 otherwise, i.e. if $T_\alpha^n x \in [\alpha, 1)$. Then, each $x \in \mathbb{T}$ yields a sequence $\psi_\alpha(x)$. Prove that any such sequence $\psi_\alpha(x)$ is uniformly recurrent as a point of the full shift. Hint: what does it mean for $\psi_\alpha(x)$ to be in a cylinder set $[w]$, i.e. to start with a certain finite string of digits? Can you characterize the set $\{x \in \mathbb{T} \ : \ \psi_\alpha(x) \in [w]\}$?

Solution: Fix any such α and x. We wish to prove that $\psi_\alpha(x)$ is uniformly recurrent. This means that for any open set U containing $\psi_\alpha(x)$, we must show that the set

$$S_U(\psi_\alpha(x)) := \{n \ : \ \sigma^n \psi_\alpha(x) \in U\}$$

has bounded gaps. Consider any such U; by definition of the product topology, U contains a cylinder set $[w]$ containing $\psi_\alpha(x)$. Clearly $S_U(\psi_\alpha(x)) \supset S_{[w]}(\psi_\alpha(x))$, and so it suffices to show that $S_{[w]}(\psi_\alpha(x))$ has bounded gaps.

Let's define I_w as the set $\{y \in \mathbb{T} \ : \ \psi_\alpha(y) \in [w]\}$, i.e. the set of points y whose orbit coding sequence begins with w. Denote by k the length of w. Then, from discussions in class, we know that

$$I_w = \bigcap_{i=0}^{k-1} T_\alpha^{-i} I_{w_i},$$

where w_i is the ith letter of w, and $I_0 = [0,\alpha)$ and $I_1 = [\alpha, 1)$ are the two coding intervals. However, this set is an intersection of intervals which are closed on the left and open on the right, and therefore is such an interval itself. The set I_w also contains x by definition (recall that $\psi_\alpha(x) \in [w]$.) So, I_w is a nonempty half-open interval, and therefore contains an open interval, let's call it J.

By minimality of $(\mathbb{T}, T_\alpha)$, the set $S_J(x) = \{n \ : \ T_\alpha^n x \in J\}$ has bounded gaps. Moreover, for any $n \in S_J(x)$,

$$T_\alpha^n x \in J \Longrightarrow T_\alpha^n x \in I_w \Longrightarrow \psi_\alpha(T_\alpha^n x) \in [w] \Longrightarrow \sigma^n \psi_\alpha(x) \in [w].$$

Therefore, $S_J(x) \subseteq S_{[w]}(\psi_\alpha(x)) \subseteq S_U(\psi_\alpha(x))$, and since $S_J(x)$ has bounded gaps, $S_U(\psi_\alpha(x))$ does as well. Since U was arbitary, this completes the proof.

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• Suppose that (X,T) is any invertible expansive topological dynamical system. I want you to construct a factor map from some two-sided symbolic system (Y,σ) to (X,T), where $Y \subseteq \{1,2,\ldots,N\}^{\mathbb{Z}}$ for some N. Here is an outline:

(a) Prove that for the expansiveness constant $\delta > 0$ of (X,T), if x,y have the property that $d(T^n x, T^n y) < \delta$ for all $n \in \mathbb{Z}$, then $x = y$.

(b) Use this to cover X with a finite collection of closed balls A_i so that, for any x, knowledge of a sequence $(k_n)_{n\in\mathbb{Z}}$ s.t. $T^n x \in A_{k_n}$ for all $n \in \mathbb{Z}$ uniquely determines x.
(c) Use this to construct a symbolic system $Y \subseteq \{1, 2, \dots, N\}^{\mathbb{Z}}$ and a factor map from (Y, σ) to (X, T). Remember that you must show that Y is closed in the product topology and σ-invariant!

Solution: (a) is clear; it's just the contrapositive of the definition of expansiveness. Then, consider the open cover $\{B_{0.4\delta}(z)\}_{z\in X}$ of X by open balls of diameter δ. By compactness of X, this has a finite subcover $\{B_{0.4\delta}(z_n)\}_{1\leq n\leq N}$, and clearly the collection of closed balls $\{\overline{B_{0.4\delta}(z_n)}\}_{1\leq n\leq N}$ covers X as well. If there exist $(n_k)_{n\in\mathbb{Z}}$ s.t. $T^k x, T^k y \in \overline{B_{0.4\delta}(z_{n_k})}$ for all $k \in \mathbb{Z}$, then clearly $d(T^k x, T^k y) < \delta$ for all n, which implies that $x = y$ by (a). Therefore, this collection of balls satisfies (b).

Now, for (c), let's define a set $Y \subseteq \{1, \dots, N\}^{\mathbb{Z}}$ as

$$Y = \{(y_k) \ : \ \exists x \in X \text{ for which } T^k x \in \overline{B_{0.4\delta}(z_{y_k})} \text{ for all } k \in \mathbb{Z}\}.$$

Define a function $\phi : Y \to X$ by setting $\phi(y)$ to be the unique x (uniqueness follows from (b)) s.t. $T^k x \in \overline{B_{0.4\delta}(z_{y_k})}$ for all $k \in \mathbb{Z}$. (Equivalently, $\bigcap_{k\in\mathbb{Z}} T^{-k}\overline{B_{0.4\delta}(z_{y_k})} = \{x\}$.)

We now claim that Y is σ-invariant and closed in the induced product topology, and that ϕ is a factor map, i.e. it is continuous and $\phi(\sigma y) = T(\phi(y))$ for all $y \in Y$. For any $y = (y_k) \in Y$, by definition $\phi(y)$ is the unique point of X so that $T^k\phi(y) \in \overline{B_{0.4\delta}(z_{y_k})}$ for all $k \in \mathbb{Z}$. Then clearly $T^k(T(\phi(y))) \in \overline{B_{0.4\delta}(z_{y_{k+1}})}$ for all $k \in \mathbb{Z}$. Then by definition of Y, the sequence $\sigma y = (y_{k+1})$ is in Y, and by definition of ϕ, $\phi(\sigma y) = T(\phi(y))$.

To see that Y is closed, consider any sequence $y^{(m)} = (y_k^{(m)})$ of sequences in Y and a limit point $y = (y_k)$ in Y so that $y^{(m)} \to y$ in the induced product topology on Y. Then the sequence $\phi(y^{(m)})$ of points in the compact space X has a limit point x. For any $n \in \mathbb{N}$, for large enough m, $y^{(m)}$ agrees with y on digits $-n$ through n. This implies that for large enough m and any $i \in [-n, n]$, $\phi(y^{(m)}) \in T^{-i}\overline{B_{0.4\delta}(z_{y_i})}$. Since T is a homeomorphism, each set $T^{-i}\overline{B_{0.4\delta}(z_{y_i})}$ is closed. Therefore, since x is a limit point of $\phi(y^{(m)})$, $x \in T^{-i}\overline{B_{0.4\delta}(z_{y_i})}$ for $i \in [-n, n]$. Since n was arbitrary, $x \in T^{-i}\overline{B_{0.4\delta}(z_{y_i})}$ for all $i \in \mathbb{Z}$, and so $(y_k) = y \in Y$. This verifies that Y is closed. In addition, this proof shows that $x = \phi(y)$. Since $\phi(y)$ is unique and our proof relied only on x being a subsequence of $\phi(y^{(m)})$, this implies that $\phi(y^{(m)}) \to x = \phi(y)$. We have therefore also verified continuity of ϕ, completing the proof.

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