MATH 4290 Homework Assignment 5 Solutions

• If (X, T) is a topological dynamical system and X is a compact metric space with metric d, prove that the quantity d_n defined by $d_n(x, y) := \max_{0 \le i < n} d(T^i x, T^i y)$ is a metric for all $n \in \mathbb{N}$.

Solution: By definition, $d_n \geq d$, and so $d_n(x, y) = 0 \implies d(x, y) = 0 \implies x = y$ since d is a metric. Similarly, $d_n(x, x) = \max_{0 \le i < n} d(T^i x, T^i x) = 0$. Symmetricity is also easy:

$$d_n(x, y) = \max_{0 \le i < n} d(T^i x, T^i y) = \max_{0 \le i < n} d(T^i y, T^i x) = d_n(y, x),$$

where the second inequality uses the fact that d is a metric.

Finally, we must check the triangle inequality:

$$\begin{aligned} d_n(x, z) = \max_{0 \le i < n} d(T^i x, T^i z) &\le \max_{0 \le i < n} (d(T^i x, T^i y) + d(T^i y, T^i z) \\ &\le \max_{0 \le i < n} d(T^i x, T^i y) + \max_{0 \le i < n} d(T^i y, T^i z) = d_n(x, y) + d_n(y, z). \end{aligned}$$

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• Prove the inequalities in Lemma 2.5.1 of the textbook.

Solution: The first inequality is proved in the book. The second inequality is $\text{span}(n, \epsilon) \le \text{sep}(n, \epsilon)$. To see this, consider a maximal ϵ-separated set S for d_n. There cannot exist x with d_n-distance at least ϵ from all points in S, as then $S \cup \{x\}$ would be a larger ϵ-separated set for d_n, a contradiction to maximality of S. Therefore, every point in X is distance less than ϵ from some point of S, i.e. S is ϵ-spanning for d_n. The minimal size of such a set is then less than or equal to $|S| = \text{sep}(n, \epsilon)$, and so $\text{span}(n, \epsilon) \le \text{sep}(n, \epsilon)$.

Finally, we must prove the third inequality: $\text{sep}(n, \epsilon) \le \text{cov}(n, \epsilon)$. To see this, consider any ϵ-separated set S for d_n and any cover $\mathcal{C}$ of X by sets of d_n-diameter less than ϵ. For every $C \in \mathcal{C}$, C contains at most one point of S; if it contained $x, y \in S$, then $d_n(x, y) \le \text{diam}(C) < \epsilon$, contradicting S being ϵ-separated. Since every $C \in \mathcal{C}$ contains at most one point of S, $|\mathcal{C}| \ge |S|$. Since $\mathcal{C}$ and S were arbitrary, this shows that the maximum size of such S is less than or equal to the minimum size of such $\mathcal{C}$, i.e. $\text{sep}(n, \epsilon) \le \text{cov}(n, \epsilon)$.

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• For the one-sided full shift $(\{0, 1\}^{\mathbb{N}}, \sigma)$, find, with proof, $\text{sep}(n, 2^{-k})$ and $\text{span}(n, 2^{-k})$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. (Reminder: the metric here is $d((x_n), (y_n)) := 2^{-\max\{n \ge 0 \,:\, x_i = y_i \ \forall i \le n\}}$.)

Solution: The following fact is immediate from definition of d and will be useful throughout this problem: for any m, $d((x_n), (y_n)) \le 2^{-m}$ iff $x_i = y_i$ for

$1 \leq i \leq m$, and $d((x_n),(y_n)) < 2^{-m}$ iff $x_i = y_i$ for $i \leq m+1$ (this is because d takes only values of the form 2^{-i}.)

Now, suppose that S is 2^{-k}-separated for d_n. Equivalently, for all $x \neq y \in S$, there exists $0 \leq i < n$ so that $d(\sigma^i x, \sigma^i y) \geq 2^{-k}$. Equivalently, there exists $0 \leq i < n$ and $1 \leq j \leq k+1$ so that $x_{i+j} \neq y_{i+j}$. Equivalently, there exists $1 \leq m \leq n+k$ so that $x_m \neq y_m$. Equivalently, the initial words of length $n+k$ in points in S are all distinct. It's clear that the maximal size of such a set is just the number of such words, i.e. $\mathrm{sep}(n, 2^{-k}) = |A|^{n+k}$.

Suppose that T is 2^{-k}-spanning for d_n. Equivalently, for every $x \in X$, there exists $t \in T$ so that $d(\sigma^i x, \sigma^i t) < 2^{-k}$ for all $0 \leq i < n$. Equivalently, for all $0 \leq i < n$ and all $1 \leq j \leq k+1$, $x_{i+j} = t_{i+j}$. Equivalently, $x_m = t_m$ for all $1 \leq m \leq n+k$. Equivalently, for every $x \in X$, there exists $t \in T$ starting with the same $n+k$-letter word as x. It's clear that the minimal size of such a set is just the number of such words, i.e. $\mathrm{span}(n, 2^{-k}) = |A|^{n+k}$.

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- If (X,T) factors onto (Y,S), prove that $h(X,T) \geq h(Y,S)$.

Solution: Choose any $\epsilon > 0$, and define ϕ the factor from (X,T) to (Y,S) and d_X, d_Y the metrics on X, Y respectively. By definition of continuity, there exists $\delta > 0$ so that $d_X(x,x') < \delta \implies d_Y(\phi x, \phi x') < \epsilon$. Then,

$$(d_X)_n(x,x') < \delta \Rightarrow \forall i \in [0,n), d_X(T^i x, T^i x') < \delta \Rightarrow \forall i \in [0,n), d_Y(\phi(T^i x), \phi(T^i x')) < \epsilon$$
$$\Rightarrow \forall i \in [0,n), d_Y(S^i \phi x), \phi(S^i \phi x')) < \epsilon \Rightarrow (d_Y)_n(\phi x, \phi x') < \epsilon.$$

Therefore, if $\mathcal{C}$ is a cover of X by sets with $(d_X)_n$-diameter less than δ, $\phi(\mathcal{C}) := \{\phi(C) \ : \ C \in \mathcal{C}\}$ is a cover of Y (by surjectivity) by sets with $(d_Y)_n$-diameter less than ϵ. By taking $\mathcal{C}$ to have minimum cardinality, we see that

$$\mathrm{cov}_Y(n,\epsilon) \leq \mathrm{cov}_X(n,\delta).$$

Taking logs, dividing by n, and letting $n \to \infty$ yields $h_\epsilon(Y,S) \leq h_\delta(X,T) \leq h(X,T)$. Letting $\epsilon \to 0$ yields $h(Y,S) \leq h(X,T)$.

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- If (X,T) is an isometry (i.e. $d(x,y) = d(Tx,Ty)$ for all $x, y \in X$), prove that $h(X,T) = 0$.

Solution: Since (X,T) is an isometry, $d(T^i x, T^i y) = d(x,y)$ for all x, y, i, and so $d_n = d$ for all d. Therefore, for every $\epsilon > 0$, $\mathrm{cov}(n,\epsilon) = \mathrm{cov}(1,\epsilon)$ for all n, meaning that

$$h_\epsilon(X,T) = \lim_{n\to\infty} \frac{\log \mathrm{cov}(n,\epsilon)}{n} = \lim_{n\to\infty} \frac{\log \mathrm{cov}(1,\epsilon)}{n} = 0.$$

So, $h(X,T) = \lim_{\epsilon \to 0^+} h_\epsilon(X,T) = 0$.

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