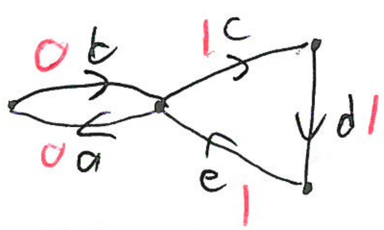
MATH 4290 Homework Assignment 6 Solutions

- Define S to be the shift with alphabet $\{0, 1\}$ consisting of all bi-infinite sequences in which the number of 0s between any closest 1s is even, and the number of 1s between any closest 0s is a multiple of 3. Prove that S is sofic and mixing, and find $h(S, \sigma)$ by using techniques similar to those used to treat the even shift in class.

Solution: Define the digraph G as in the figure (edges named in black) with labeling ℓ given by $a, b \mapsto 0$ and $c, d, e \mapsto 1$ (in red).



Then $X(G)$ is an edge shift, with transition matrix $M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$.

Since $M^7 = \begin{pmatrix} 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix}$ has all positive entries, $X(G)$ is mixing. Clearly ℓ is a letter-to-letter factor map on $X(G)$, and so $\ell(X(G)) = X(G, \ell)$ is sofic (it's the factor of an SFT) and mixing (factors of mixing systems are mixing). It is also fairly clear that $S = X(G, \ell)$; the fact that 0s occur only in a cycle of length 2 and 1s occur only in a cycle of length 3 mean that runs of 0s must have even length and runs of 1s must have length a multiple of 3, and it is easy to construct paths on G labeled by any point in S.

It remains only to find $h(S)$, which we will do by showing that $h(S) = h(X(G))$ and then finding $h(X(G))$ with the Perron-Frobenius Theorem. To see the relationship between $h(S)$ and $h(X(G))$, we need to find the relationship between $c_n(S)$ and $c_n(X(G))$. Since ℓ is a letter-to-letter factor map from $X(G)$ onto S, ℓ also induces a surjection from $L_n(X(G))$ to $L_n(S)$. In addition, this surjection is nearly one-to-one: if you're looking for a ℓ-preimage of any word in $L_n(S)$ containing both a 0 and a 1, then that preimage is uniquely determined. This is because 10 has unique preimage ea, 01 has unique preimage bc, and once you know one letter in the preimage of a word w, the rest of the letters are forced. (This is because of the "left/right-resolving" properties we discussed in class; no vertex of G has two outgoing edges with the same label.) All of this means that the only words with multiple preimages are 0^n and 1^n. There are two preimages of 0^n, namely $abab\ldots$ and $baba\ldots$, and three preimages of 1^n, namely $cdecde\ldots$, $decdecd\ldots$, and $ecdecd\ldots$. Therefore, $|L_n(S)| = |L_n(X(G))| - 3$.

Finally, as discussed in class, $h(X(G)) = \log \lambda$ for the Perron eigenvalue of M. The characteristic polynomial of M is $\det(M - Ix) = -x^5 + x^3 + x^2 = -x^2(x^3 - x - 1)$. The roots of this are 0 and the three roots of $x^3 - x - 1$; the largest real root is

$$\lambda = \left(\frac{9+\sqrt{69}}{18}\right)^{1/3} + \left(\frac{9-\sqrt{69}}{18}\right)^{1/3} \approx 1.3247.$$

This means that $c_n(X(G))$ grows roughly like $e^{nh(X(G))} = e^{n \log \lambda} = \lambda^n$, and so $c_n(S) = c_n(X(G)) - 3$ grows with the same exponential rate, i.e. $h(S) = \log \lambda$.

∎

- Define $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, $X = A^{\mathbb{N}}$, and the metric d on X by

$$d((x_n), (y_n)) = \sum_{n \in \mathbb{N}} 2^{-n} |x_n - y_n|,$$

where $|x_n - y_n|$ is defined by the usual absolute value in $\mathbb{R}$. (You do NOT have to show that d is a metric.) If σ is the usual left shift on X, prove that $h(X, \sigma) = \infty$.

Solution: Choose any $k \in \mathbb{N}$; we will give lower bounds on $\mathrm{sep}(n, \frac{1}{2k^2})$ for all n. Consider the set S_n of sequences in X whose first n coordinates are from $\{1, \frac{1}{2}, \ldots, \frac{1}{k}\}$ and whose remaining coordinates are 0. We claim that S_n is $(d_n, \frac{1}{k^2})$-separated. To see this, consider $s \neq s' \in S_n$. They differ on one of the first n coordinates, say the ith for $i \leq n$. Then, $\sigma^{i-1}s$ and $\sigma^{i-1}s'$ differ on the first coordinate, meaning that $d(\sigma^{i-1}s, \sigma^i s') \geq 2^{-1}\left|\frac{1}{k} - \frac{1}{k-1}\right| > \frac{1}{2k^2}$. Therefore, $d_n(s, s') > \frac{1}{2k^2}$ as well, verifying the desired separation.

Since $\mathrm{sep}(n, \frac{1}{2k^2})$ is the maximum size of a $(d_n, \frac{1}{k^2})$-separated set, $\mathrm{sep}(n, \frac{1}{2k^2}) \geq |S_n| = k^n$. Therefore,

$$h_{\frac{1}{2k^2}} = \lim_{n\to\infty} \frac{\log \mathrm{sep}(n, \frac{1}{2k^2})}{n} \geq \limsup_{n\to\infty} \frac{\log k^n}{n} = \log k.$$

This means that

$$h(X, \sigma) = \lim_{\epsilon \to 0^+} h_\epsilon = \lim_{k\to\infty} h_{\frac{1}{2k^2}} = \infty.$$

∎

- Prove that if X is a mixing 1-step SFT (i.e. the forbidden list is given by pairs of adjacent letters) and $|X| > 1$, then $h(X, \sigma) > 0$.

Solution: There are several solutions to this problem. I'll present one here requiring no linear algebra. Since $|X| > 1$, there are at least two letters appearing in points of X, call them a, b. Since X is mixing, there exists n so that $[a] \cap \sigma^{-n}[a]$, $[a] \cap \sigma^{-n}[b]$, $[b] \cap \sigma^{-n}[a]$ are all nonempty, i.e. there exist $(n+1)$-letter words in $L(X)$ of the form aua, avb, bwa. Since X is 1-step, the $(2n+1)$-letter words $auaua$ and $avbwa$ are both in $L(X)$ as well (all adjacent pairs in these words are part of either aua, avb, or bwa, all of which are known to be legal in X).

Using the same logic, any word obtained by gluing together copies of $auaua$ and $avbwa$ at their beginning/ending a is also in $L(X)$, e.g. words like

$$auauavbwavbwauauauaua.$$

For any k, this yields 2^k words (from all choices of $auaua$ or $avbwa$ at each location) of length $2kn+1$. Therefore, $c_{2kn+1}(X) \geq 2^k$ for all k, meaning that

$$h(X,\sigma) = \lim_{m\to\infty} \frac{\log c_m(X)}{m} = \lim_{k\to\infty} \frac{\log c_{2kn+1}(X)}{2kn+1} \geq \limsup_{k\to\infty} \frac{\log 2^k}{2kn+1}$$
$$= \limsup_{k\to\infty} \frac{k \log 2}{2kn+1} = \frac{1}{2n} > 0.$$

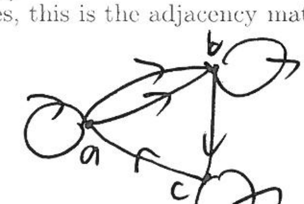
∎

• Define a directed graph G so that the associated edge shift $X(G)$ is mixing and has $h(X(G),\sigma) = \log(\sqrt[3]{2}+1)$.

Solution: We'll work backwards. We know that $h(X(G)) = \log\lambda$ for the Perron eigenvalue λ, so we want

$$\lambda = \sqrt[3]{2}+1 \Longrightarrow \lambda - 1 = \sqrt[3]{2} \Longrightarrow (\lambda-1)^3 = 2 \Longrightarrow (1-\lambda)^3 + 2 = 0.$$

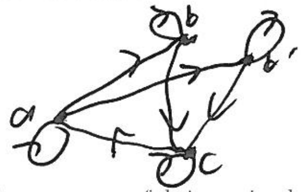
So, we want our characteristic polynomial to look like (or at least to contain as a factor) $(1-x)^3+2$. This looks kind of like a characteristic polynomial, since it has several $1-x$ factors. In fact, it's easy to show that $(1-x)^3+2 = \det(M - Ix)$ for $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Forgetting for a moment the restriction on multiple edges between vertices, this is the adjacency matrix for the following digraph:



What's more, $M^2 = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, which has all positive entries, so our $X(G)$ is mixing. We need to check that $\lambda = \sqrt[3]{2}+1$ is actually the largest real eigenvalue, but this is simple: the roots of $(1-x)^3+2=0$ are given by 1 minus the three possible cube roots of -2, and the other two will be imaginary numbers. So, λ is the only real root of $\det(M-Ix)$, and we are done.

This is a valid answer to the problem, since in fact everything we did in class works equally well for graphs with multiple edges between a pair of vertices.

However, it is also possible to do this without such edges. Suppose that the vertex b is "split" into copies b and b' as follows, creating a new graph G':



Now, whichever copy of b is arrived at from a, its outgoing options are the same as b from the original graph. This should mean that the edge shifts have the same entropy, and in fact this is true. The adjacency matrix of G' is $M' = \begin{pmatrix} 1&1&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 1&0&0&1 \end{pmatrix}$, with characteristic polynomial $\det(M' - Ix) = (1-x)^4 + 2(1-x) = (1-x)[(1-x)^3 + 2]$, with the same roots as before, along with 1. Then $\lambda = \sqrt[3]{2} + 1$ is still the largest eigenvalue, and so $h(X(G')) = \log \lambda$.

∎

• Prove that it is impossible to find G as in the previous problem with $h(X(G)), \sigma) = \log(3 - \sqrt{2})$.

Solution: If such an $X(G)$ existed, then $\lambda = 3 - \sqrt{2}$ would be the largest real eigenvalue of the characteristic polynomial of the adjacency matrix M of G. Suppose there is such a G, and denote this characteristic polynomial by $p(x)$. We claim that $3 + \sqrt{2}$ must also be a root of $p(x)$, meaning that λ was not the largest real eigenvalue, and deriving a contradiction.

This is provable via Galois theory, but there are much simpler arguments also. For instance, note that for any n, $(3 + \sqrt{2})^n + (3 - \sqrt{2})^n$ is an integer (just use the binomial theorem for this; the terms with odd powers of $\sqrt{2}$ cancel, and even powers of $\sqrt{2}$ are integers). Therefore, since p has integer coefficients, $p(3 + \sqrt{2}) + p(3 - \sqrt{2}) = p(3 + \sqrt{2})$ is an integer. However, by similar reasoning, $(3-\sqrt{2})^n - (3-\sqrt{2})^n$ is an integer times $\sqrt{2}$ (again, this comes from the binomial theorem, basically the opposite of the previous reasoning). Again, since p has integer coefficients, $p(3 + \sqrt{2}) - p(3 - \sqrt{2}) = p(3 + \sqrt{2})$ is an integer times $\sqrt{2}$. However, since $\sqrt{2} \notin \mathbb{Q}$ and $p(3+\sqrt{2})$ is both an integer and an integer multiple of $\sqrt{2}$, it must be 0. So, $3 + \sqrt{2}$ is also a root of $p(x)$, meaning that $3 - \sqrt{2}$ is not the largest real eigenvalue of M, a contradiction.

∎