MATH 4290 Homework Assignment 7

Due on Thursday, November 8th, at the BEGINNING of class.

• If $x = \ldots 10101010.234234234\ldots \in \{0,1,2,3,4\}^{\mathbb{Z}}$, and we define the topological dynamical system (X, σ) with $X = \overline{O(x)} \subseteq \{0,1,2,3,4\}^{\mathbb{Z}}$ (here σ is, as always, the left shift), give a complete description of all σ-invariant probability measures on the Borel σ-algebra $B(X)$. Which of these are ergodic?

• Show that for any measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$, if we define $Z = A \setminus (\bigcup_{n=1}^{\infty} T^{-n}A)$, then all of the sets Z, $T^{-1}Z$, $T^{-2}Z$, $\ldots$ are pairwise disjoint.

• For an invertible measure-preserving dynamical system $(X, \mathcal{P}(X), \mu, T)$ with X finite, show that μ is ergodic if and only if it is distributed equally over a single periodic orbit, i.e. $\mu = \frac{1}{n}\delta_x + \ldots + \frac{1}{n}\delta_{T^{n-1}x}$ for some x, n with $T^n x = x$.

• For $(X, \mathcal{B}, \mu, T)$ with T invertible, show that $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if the following statement is true: for every $A \in \mathcal{B}$ with $\mu(A) > 0$, it is the case that $\mu\left(\bigcup_{n \in \mathbb{Z}} T^n A\right) = 1$.

• Show the following extension of the Poincaré recurrence theorem: for any measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$, it is the case that for μ-almost every $x \in A$, there exist infinitely many $n \in \mathbb{N}$ for which $T^n x \in A$.