• Show that there exists a factor map (called a topological semiconjugacy in Brin and Stuck) from \((\mathbb{T}, R_\alpha)\) to \((\mathbb{T}, R_{2\alpha})\) for any \(\alpha \in \mathbb{T}\). (Here, as usual, \(R_\alpha\) denotes the circle rotation by \(\alpha\).)

• Show that for any \(\alpha \notin \mathbb{Q}\) and \(r \in \mathbb{Q}\), there does not exist a factor map from \((\mathbb{T}, R_\alpha)\) to \((\mathbb{T}, R_r)\).

• Show that for any \(\alpha \notin \mathbb{Q}\), there does not exist a factor map from \((\mathbb{T}, R_{2\alpha})\) to \((\mathbb{T}, R_\alpha)\). (Hint: the image of a connected set under a continuous map is connected...)

• Show that the map \(\phi : \{0, 1\}^N \rightarrow [0,1)\) defined by \(\phi(x_1,x_2,\ldots) = \sum_{n=1}^{\infty} x_i 2^{-i}\) is continuous, with the product topology on \(\{0, 1\}^N\) and the usual (Borel) topology on \([0,1)\).

• Show that if \(\phi\) is a factor map from \((X, T)\) to \((Y, S)\) and if \((X, T)\) is topologically transitive, then \((Y, S)\) is topologically transitive.

• For any two topological dynamical systems \((X, T)\) and \((Y, S)\), define the product system \((X \times Y, T \times S)\), where \(T \times S : (x, y) \mapsto (Tx, Sy)\). Show that

  (i) The product of two topologically transitive systems is not necessarily topologically transitive
  (ii) The product of two topologically mixing systems must be topologically mixing.