For a measure-preserving dynamical system \((X, B, \mu, T)\), show that \((X \times X, \sigma(B \times B), \mu \times \mu, T \times T)\) is ergodic if and only if \((X \times X \times X, \sigma(B \times B \times B), \mu \times \mu \times \mu, T \times T \times T)\) is ergodic. (Here, \(\sigma(A)\) is the \(\sigma\)-algebra generated by an algebra \(A\). The notation just means that products of measurable sets generate the \(\sigma\)-algebras of measurable sets for \(\mu \times \mu\) and \(\mu \times \mu \times \mu\).)

Prove that if \((X, B, \mu, T)\) is a measure-preserving dynamical system and \(f, g \in L^2(X)\) are eigenfunctions for distinct eigenvalues \(\lambda \neq \eta\) respectively, then \(\langle f, g \rangle = 0\), where \(\langle \cdot, \cdot \rangle\) is the usual inner product in \(L^2(X)\).

Prove that if \((X, B, \mu, T)\) is (measure-theoretically) mixing and \(\mu(U) > 0\) for every nonempty open set \(U\), then \((X, T)\) is topologically mixing.

Based on your last homework problem from last week, prove that for any irrational \(\alpha\), Lebesgue measure is the only invariant Borel measure for the rotation \(R_\alpha\) on \(\mathbb{T}\). (Hint: if there’s another measure \(\nu\), think about what the ergodic theorem says about visits of \(\nu\)-a.e. point to an interval \(I\)...)

Prove that if \((X, B, \mu, T)\) is weak mixing, then for any \(n \in \mathbb{Z}\), \((X, B, \mu, T^n)\) is also weak mixing.

Show that if \((X, B, \mu, T)\) and \((X, B, \nu, T)\) are both measure-preserving dynamical systems (i.e. \(\mu\) and \(\nu\) are both invariant measures on the same space for the same transformation), and if \(\mu \neq \nu\), then the measure \(\frac{1}{2}(\mu + \nu)\) is NOT ergodic. (Hint: a set \(A\) with \(\frac{1}{2}(\mu + \nu)(A) = 1\) must have \(\mu(A) = 1\) and \(\nu(A) = 1\).)