FSEM Homework Assignment 4 Solutions

1. Recall our method for filling in entries of this matrix: for any choice of wagers for Rose and Colin, we consider all four possible (random) possibilities for whether they get the Final Jeopardy! question right or wrong. For example, if Rose wagers 1 and Colin wagers 2, then there are four possible outcomes:

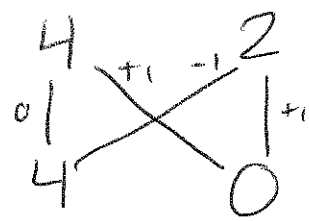
Rose right, Colin right: Rose has $3 + 1 = 4$, Colin has $2 + 2 = 4$, they tie and payoff is 0.

Rose right, Colin wrong: Rose has $3 + 1 = 4$, Colin has $2 - 2 = 0$, Rose wins, payoff $+1$.

Rose wrong, Colin right: Rose has $3 - 1 = 2$, Colin has $2 + 2 = 4$, Colin wins, payoff -1.

Rose wrong, Colin wrong: Rose has $3 - 1 = 2$, Colin has $2 - 2 = 0$, Rose wins, payoff 0.

So, we have four possible outcomes, each of which happens $\frac{1}{4}$ of the time. The expected payoff associated to these wagers is then $0(\frac{1}{4}) + (1)(\frac{1}{4}) + (-1)(\frac{1}{4}) + (1)(\frac{1}{4}) = \frac{1}{4}$.



$$\frac{0+1-1+1}{4} = \boxed{\frac{1}{4}}$$

The other eleven entries can be filled in the same way, leading to the following matrix of expected payoffs:

	0	1	2
0	1	$\frac{1}{2}$	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
2	0	$\frac{1}{4}$	$\frac{1}{2}$
3	0	0	$\frac{1}{4}$

Note that Rose 3 is dominated by Rose 2, so we reduce to the following 3×3 game:

	0	1	2
0	1	$\frac{1}{2}$	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
2	0	$\frac{1}{4}$	$\frac{1}{2}$

This has no saddle points and no dominated strategies, so we have to solve with mixed strategies. Let's try to find Rose's optimal mixed strategy. Say that x

represents the probability that Rose bets 0, y represents the probability that Rose bets 1, and then $1-x-y$ will be the probability that Rose bets 2.

If Colin counters with 0, the expected payoff is $1(x)+\frac{1}{2}(y)+0(1-x-y) = x+\frac{y}{2}$.
If Colin counters with 1, the expected payoff is $\frac{1}{2}(x)+\frac{1}{2}(y)+\frac{1}{4}(1-x-y) = \frac{1}{4}+\frac{x}{4}+\frac{y}{4}$.
If Colin counters with 2, the expected payoff is $0(x)+\frac{1}{4}(y)+\frac{1}{2}(1-x-y) = \frac{1}{2}-\frac{x}{2}-\frac{y}{4}$.

To use the method of equalizing expectations, Rose wants these three to be equal. So $x+\frac{y}{2} = \frac{1}{4}-\frac{x}{4}-\frac{y}{4}$ and $x+\frac{y}{2} = \frac{1}{2}-\frac{x}{2}-\frac{y}{4}$. It's easiest if we multiply both sides of both equations by 4 to remove the fractions. Our system of equations is then $4x+2y = 1+x+y$ and $4x+2y = 2-2x-y$.

$4x+2y = 1+x+y \Longrightarrow 3x = 1+y \Longrightarrow y = 1-3x$. Plug this into the second equation:

$4x+2(1-3x) = 2-2x-(1-3x) \Longrightarrow 4x+2-6x = 2-2x-1+3x \Longrightarrow -2x+2 = 1+x \Longrightarrow 3x = 1 \Longrightarrow x = \frac{1}{3}$.

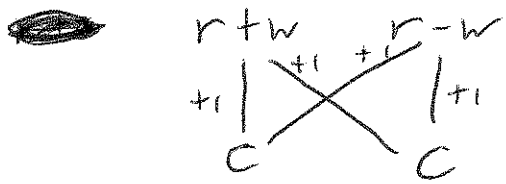
Plug this into our formula for y: $y = 1-3x$, so $y = 1-3(\frac{1}{3}) = 0$.
So Rose's optimal mixed strategy is to wager 0 one-third of the time and 2 two-thirds of the time. To see what payoff this guarantees, plug x and y in to any of the three formulas describing the expected payoffs for Colin's counterstrategies, for instance $x+\frac{y}{2}$: $\frac{1}{3}+2(0) = \frac{1}{3}$. So, Rose's mixed strategy guarantees a payoff of $\frac{1}{3}$ regardless of what Colin does.

I will not repeat all of the work here, but similar work shows that Colin's optimal mixed strategy is the same, and guarantees a payoff of $\frac{1}{3}$ regardless of Rose's strategy. Therefore, the value of this game is $\frac{1}{3}$.

2. From the examples we did in class (7 vs. 5, 7 vs. 4, 3 vs. 1, 1000 vs 999), it looks like the $\frac{1}{2}$ in the first column always shows up when Rose wagers $r-c$, the difference between the two players' starting amounts. Notice that we're in the first column, and Colin's wager is therefore always 0. Then, our guess is that, if we call Rose's wager w:

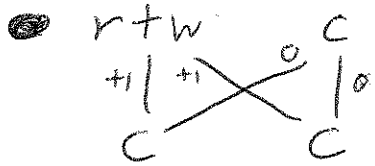
- The expected payoff is 1 if $w < r-c$
- The expected payoff is $\frac{1}{2}$ if $w = r-c$
- The expected payoff is 0 if $w > r-c$

Let's check these. If $w < r - c$, then $w + c < r$, and so $c < r - w$. Therefore, Rose always wins: the least she could have after answering her question is $r - w$ (if she's wrong), which is still greater than c. Colin wagered 0, so he has c dollars after answering her question no matter what, and Rose will always have more. So, her expected payoff is 1.



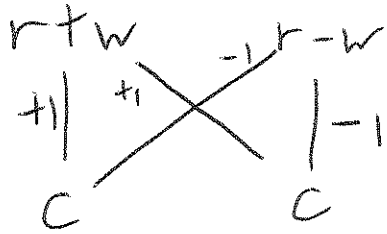
$$\frac{1+1+1+1}{4} = \boxed{1}$$

If $w = r - c$, then $r - w = r - (r - c) = c$. So, Rose will be tied with Colin if she gets the question wrong, and will beat him if she gets the question right. Her expected payoff is then $\frac{1}{2}$.



$$\frac{1+1+0+0}{4} = \boxed{\frac{1}{2}}$$

If $w > r - c$, then $w + c > r$, and so $c > r - w$. Therefore, Rose will lose if she gets the question wrong, since she will have $r - w$ dollars and Colin will have c dollars. She still wins if she gets the question right, since she started off with more money than Colin and gained more. Her expected payoff is then 0.



$$\frac{1+1+(-1)+(-1)}{4} = \boxed{0}$$