A CHARACTERIZATION OF THE SETS OF PERIODS WITHIN SHIFTS OF FINITE TYPE

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Abstract. In this paper, we characterize precisely the possible sets of periods and least periods for the periodic points of a shift of finite type (SFT). We prove that a set is the set of least periods of some mixing SFT iff it is either $\{1\}$ or cofinite, and the set of periods of some mixing SFT iff it is cofinite and closed under multiplication by arbitrary natural numbers. We then use these results to derive similar characterizations for the class of irreducible SFTs and the class of all SFTs. Specifically, a set is the set of (least) periods for some irreducible SFT iff it can be written as a natural number times the set of (least) periods for some mixing SFT, and a set is the set of (least) periods for an SFT iff it can be written as the finite union of the sets of (least) periods for some irreducible SFTs.

1. Introduction

The results in this paper are about dynamical systems. Modern dynamical systems theory has a relatively short history, though scientists from many disciplines have begun to use nonlinear dynamics techniques to describe problems ranging from physics and chemistry to ecology and economics. Fundamentally, a dynamical system is a set or space with structure, usually denoted by X, partnered with a function or map, usually denoted by f, that preserves that structure through repeated iterations. This function f can then be applied arbitrarily many times to subsets or elements of X, which incites certain possible patterns. One of the simplest is when a point returns to itself after some number (say n) of iterations of f; such a point is said to be periodic with period n. Different points of the system can have different periods, and so a simple natural object of study is the set of periods of points of a given dynamical system. The celebrated Sharkovsky's Theorem gave some surprising information about this set of periods for dynamical systems given by continuous self-maps of intervals.

Sharkovsky's Theorem ([6]). For any interval I in $\mathbb{R}$, if $f : I \to I$ is continuous and has a point of least period k, then there exist points of all least periods less than k in the Sharkovsky ordering, where the ordering is as follows:

$3 > 5 > 7 > 9 > 11...$
$> 6 > 10 > 14 > 18 > 22...$

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$> 12 > 20 > 28 > 36 > 44...$
...
$... > 16 > 8 > 4 > 2 > 1.$

In particular, Sharkovsky's Theorem implies that for any such f, the set of natural numbers which are least periods of periodic points for f is a downward closed set with respect to the Sharkovsky ordering. In fact, examples are also constructed in [6] which, given any such (nonempty) downward closed set, yield f which realizes that set as the least periods of periodic points. This then yields a complete characterization of which sets can appear as the sets of least periods for such f. The goal of the present work is to obtain such a characterization for a completely different class of dynamical systems, called the shifts of finite type.

Here we step into the realm of symbolic dynamics. For symbolic dynamical systems, one begins with a finite set of symbols called the alphabet and denoted by A. Elements of A are called letters and can be combined to form "words" or "blocks." A symbolic dynamical system, or shift space, is a subset of all possible bi-infinite sequences created with the alphabet A based on a collection of "forbidden blocks" $\mathcal{F}$, essentially rules on what words or symbols can and cannot appear in these bi-infinite sequences. For shift spaces, the dynamics are always given by the shift map σ, which shifts a sequence in the space one unit to the left. A shift space described by a finite set of forbidden blocks is called a shift of finite type (SFT). An example of an SFT would be where X is the set of all binary sequences with no two 1s next to each other, induced by $\mathcal{F} = \{11\}$. This is known as the golden mean shift. Because they have a simple representation using a finite, directed graph (see Section 3), SFTs are attractive to study, as questions about the SFT can typically be phrased as questions about the graph which can be translated back to the original shift.

A periodic point of a shift space is just a bi-infinite sequence made only of a word w of length p repeated bi-infinitely with no additional words, which is then said to have period p. In this work, we study periodic points in SFTs as, though they are in some sense the "simplest" shifts, they play an important role in dynamical systems by facilitating the study of more complex systems. The main results of this work are a characterization for shifts of finite type analogous to Sharkovsky's Theorem, along with a corresponding characterization for the sets of (not necessarily least) periods for shifts of finite type. Unlike the $f : I \to I$ case above, our characterizations do not come from any ordering of $\mathbb{N}$, but rather from structural properties of the sets.

Theorem 1.1. *A set S is closed under $\mathbb{N}$-multiples and cofinite if and only if $\exists$ a topologically mixing SFT such that S is the set of periods of its periodic points.*

Theorem 1.2. *A set R can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set which is closed under $\mathbb{N}$-multiples, if and only if $\exists$ a topologically transitive SFT such that R is the set of periods of its periodic points.*

Theorem 1.3. *A set Q can be written as $\bigcup_{i=1}^{n} p_i \cdot S_i$ for some $p_i \in \mathbb{N}$ and cofinite sets S_i which are closed under $\mathbb{N}$-multiples if and only if $\exists$ an SFT such that Q is the set of periods of its periodic points.*

Theorem 1.4. *A set S is either $\{1\}$ or cofinite if and only if $\exists$ a topologically mixing SFT such that S is the set of least periods of its periodic points.*

Theorem 1.5. *A set R is either a singleton or can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set, if and only if $\exists$ a topologically transitive SFT such that R is the set of least periods of its periodic points.*

Theorem 1.6. *A set Q can be written as $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$ for some finite set U, $p_i \in \mathbb{N}$, and cofinite sets S_i, if and only if $\exists$ an SFT such that Q is the set of least periods of its periodic points.*

In addition to the relation to Sharkovsky's theorem already outlined, these results also connect to other characterizations of various other important objects for SFTs, most notably topological entropy ([4]) and the Artin-Mazur zeta function ([3]). The latter is most relevant to our work, due to the connection of the zeta function to periodic points. The zeta function is a formal power series defined by

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} p_n \frac{z^n}{n}\right),$$

where p_n is the number of points of period n in the system. For SFTs, the zeta-function always has the form $\frac{1}{p(z)}$ for some polynomial p (see [1]), and the classification from [3] is in terms of these $p(z)$, more specifically in terms of the sets of non-zero complex numbers (with multiplicity) which can be realized as the roots of such $p(z)$. Relevant for this work is the fact that knowledge of the zeta function is theoretically equivalent to knowledge of p_n for all n, and the set of periods is the set of exponents with positive coefficients. Therefore, theoretically speaking, the classification from [3] contains enough information to derive a classification of the sets of periods for SFTs. However, practically speaking, it is not at all simple to turn information about roots of $p(z)$ into information about the set of exponents with positive coefficients for the power series expansion of $\frac{1}{p(z)}$.

Finally, we note that the possible sets for a generalized notion of least periods for multidimensional SFTs (which consist of $\mathbb{Z}^d$-indexed arrays of letters rather than sequences) were recently characterized in [2]. As is often the case for multidimensional SFTs, their characterization is in recursion-theoretic terms and much more complicated than the ones we derive in one dimension. It is noted in [2] that the set of least periods for a (one-dimensional) SFT must be semi-linear, and this is true. However, as our results show, not all semi-linear sets are realizable in this way; for instance, the set of positive odd integers is not the set of least periods of any (one-dimensional) SFT. It is strange that the much more complicated and difficult results of [2] appeared even though the one-dimensional characterization does not seem to be present anywhere in the literature; we hope that our results fill this gap.

2. Definitions

Definition 1. *A* ***topological dynamical system*** *is a pair (X, T) where X is a compact metric space and $T : X \to X$ is a continuous map.*

Definition 2. *For any finite set of symbols, A (which we call an alphabet), the* ***full A-shift*** *is the collection $A^{\mathbb{Z}} = [x = (x_i)_{i\in\mathbb{Z}} : x_i \in A$ for all $i \in \mathbb{Z}]$ of all bi-infinite sequences of symbols from A.*

Definition 3. *The* ***shift map*** *σ on the full shift $A^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate y_i is x_{i+1}, the $(i+1)$th coordinate of x.*

Definition 4. *A point x is* ***periodic*** *for σ if $\sigma^n(x) = x$ for some $n \geq 1$. x is said to have period n under σ.*

Definition 5. *For a point x that is periodic, the smallest positive integer n for which $\sigma^n(x) = x$ is the* ***least period*** *of x.*

Definition 6. *Let (X, T) and (Y, S) be two topological dynamical systems. (X, T) and (Y, S) are* ***conjugate*** *if there exists between them a homeomorphism $h : X \to Y$ such that $h(T(x)) = S(h(x))$ for all $x \in X$.*

We note that if (X, T) and (Y, S) are conjugate via $h : X \to Y$, then $T^n x = x$ iff $S^n h(x) = h(x)$, and so conjugacy preserves the number of points of (least) period n in any dynamical system.

Definition 7. *A* ***shift space*** *is a subset X of a full shift $A^{\mathbb{Z}}$ such that for some collection $\mathcal{F}$ of forbidden blocks over A, $X = X(\mathcal{F})$, the set of all possible bi-infinite sequences that do not contain any blocks from $\mathcal{F}$.*

Whenever X is a shift space, (X, σ) is a topological dynamical system when X is given the induced product topology from $A^{\mathbb{Z}}$.

Definition 8. *A* ***shift of finite type*** *is a shift space equal to $X(\mathcal{F})$ for some finite collection of forbidden blocks.*

Definition 9. *The* ***language*** *is the set of all possible blocks of length $n \in \mathbb{N}$ of a shift space X, denoted $B(X)$.*

Definition 10. *A shift space X is* ***irreducible*** *if for every ordered pair of blocks $u, v \in B(X)$ there exists $w \in B(X)$ such that $uwv \in B(X)$.*

Definition 11. *A shift space X is* ***mixing*** *if, for every ordered pair $u, v \in B(X)$, there is an N such that for each $n \geq N$ there exists $w \in B_n(X)$ such that $uwv \in B(X)$.*

Definition 12. *A* ***graph*** *G consists of a finite set $V = V(G)$ of vertices (or states) together with a finite set $E = E(G)$ of edges. All of the graphs discussed in this paper are directed graphs, meaning that each edge goes from one vertex to another, called the initial and terminal vertices of the edge, respectively.*

Definition 13. *A cycle C of an arbitrary graph G is called* ***non-elementary*** *if C is comprised of a single smaller cycle followed two or more times. A cycle is called* ***elementary*** *if it is not non-elementary.*

Definition 14. *A graph G is* ***irreducible*** *if for every ordered pair of vertices I and J there is a path in G starting at I and terminating at J.*

Definition 15. *Given an irreducible graph G, the* ***period of*** *G, denoted $per(G)$, is the greatest common divisor of its cycle lengths.*

Definition 16. *An irreducible graph is* ***aperiodic*** *if $per(G) = 1$.*

Definition 17. *A graph is* ***primitive*** *if it is irreducible and aperiodic.*

Definition 18. *A set S is* ***closed under $\mathbb{N}$-multiples*** *if for all $n \in S$, mn is also in S for all $m \in \mathbb{N}$.*

3. Preliminaries

The following theorems, definitions, and descriptions are used extensively in the proofs of our results. We will see that any SFT can be studied by way of an associated graph, every graph can be broken down into primitive pieces, and from these primitive graphs we can build our results.

Definition 19. *For an arbitrary graph G with set of edges $E(G)$, the* ***edge shift*** *χ_G, is the shift space over the alphabet $A = E(G)$ consisting of all bi-infinite sequences of edges which are connected end-to-end in G.*

By Proposition 2.2.6 from [5], for any graph G, the associated edge shift χ_G is an SFT. Surprisingly, every SFT can also be depicted as a graph.

Proposition 3.1. *For any SFT X, there exists a graph G such that X is conjugate to the edge shift χ_G. In addition, if X is transitive, then G can be taken to be irreducible, and if X is mixing, then G can be taken to be primitive.*

Proof. The first sentence follows from Theorem 2.3.2 from [5]. The reader may check that the remaining statements hold for the construction done there. □

This then allows a connection to be made between the periodic points of an SFT and the cycle lengths of its associated graph G:

Proposition 3.2. *For any SFT X, there exists a graph G such that for all $p \in \mathbb{Z}$, the number of points of (least) period p in X equals the number of (elementary) cycles of length p in G.*

Proposition 3.3. *For any topologically transitive SFT X, there exists an irreducible graph G such that for all $p \in \mathbb{Z}$, the number of points of (least) period p in X equals the number of (elementary) cycles of length p in G.*

Proposition 3.4. *For any topologically mixing SFT X, there exists a primitive graph G such that for all $p \in \mathbb{Z}$, the number of points of (least) period p in X equals the number of (elementary) cycles of length p in G.*

Proof. By Proposition 3.1, for any SFT X we can find a graph G such that $X \cong \chi_G$. Then by Proposition 2.2.12 from [5], the number of cycles of length m in G is equal to the number of points in the edge shift χ_G with period m. To prove this, Lind and Marcus use an object called the adjacency matrix, but we will not need this object in our work. X is conjugate to χ_G, and as we defined, conjugacy preserves periodic points, completing the proof. The proofs of Propositions 3.3 and 3.4 follow similarly. □

Thus, in light of Propositions 3.1 - 3.4, the following six theorems are equivalent to Theorems 1.1 - 1.6:

Theorem 3.5. *A set S is closed under $\mathbb{N}$-multiples and cofinite if and only if $\exists$ a primitive graph G such that S is the set of cycle lengths in G.*

Theorem 3.6. *A set R can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set which is closed under $\mathbb{N}$-multiples, if and only if $\exists$ an irreducible graph H such that R is the set of cycle lengths in H.*

Theorem 3.7. *A set Q can be written as $\bigcup_{i=1}^{n} p_i \cdot S_i$ for some $p_i \in \mathbb{N}$ and cofinite sets S_i which are closed under $\mathbb{N}$-multiples if and only if $\exists$ a graph F such that Q is the set of cycle lengths in F.*

Theorem 3.8. *A set S is either $\{1\}$ or cofinite if and only if $\exists$ a primitive graph G such that S is the set of elementary cycle lengths in G.*

Theorem 3.9. *A set R is either a singleton or can be written as $p \cdot S$, where $p \in \mathbb{N}$ and S is a cofinite set, if and only if $\exists$ an irreducible graph H such that R is the set of elementary cycle lengths in H.*

Theorem 3.10. *A set Q can be written as $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$ for some finite set U, $p_i \in \mathbb{N}$, and cofinite sets S_i, if and only if $\exists$ a graph F such that Q is the set of elementary cycle lengths in F.*

Now, the graphs themselves can be decomposed into irreducible and primitive components, the consequences of which will be used extensively in the proofs of our results.

Proposition 3.11. *For every graph G, there exist irreducible subgraphs $G_1, G_2, ..., G_k$ such that the set of (elementary) cycles that appear in G is the disjoint union of the sets of (elementary) cycles that appear in the G_i.*

Proof. Begin by separating G into communicating classes $C_i \subset V(G)$ defined by the collections of vertices such that for each pair of vertices I and J within a collection, there exists a path I to J and J to I. Let G_i be the subgraph $G|_{C_i}$. We claim that no cycles of G can contain vertices from two different communicating classes. To see this, let C, B be two communicating classes. Then suppose for a contradiction there exists an edge connecting a vertex of C to a vertex of B and another edge connecting a vertex of B to a vertex of C. This would create a larger communicating class, which is a contradiction by definition. By again considering the definition of communicating classes, we see the G_i are irreducible. It is clear that every cycle of G is part of some G_i, thus the set of cycles that appear in G is the disjoint union of the sets of cycles that appear in the G_i. □

Proposition 3.12. *Any irreducible graph G has an associated primitive graph G' for which the set of (elementary) cycles of G is $p \cdot S$, where $p = per(G)$ and S is the set of (elementary) cycle lengths of G'.*

Proof. Let G be irreducible. By Proposition 4.5.6 from [5], V(G) can be grouped into exactly p period classes which can be ordered as $D_0, D_1, ..., D_{p-1}$ so that every edge that starts in D_i terminates in D_{i+1} (or in D_0 if $i = p-1$). The comment following Proposition 4.5.6 states that there is an associated graph G^p, called the higher power graph, that consists of p primitive (aperiodic and irreducible), disjoint subgraphs $G_1, ..., G_p$. In Exercise 4.5.6 from [5], it is shown that the edge shifts X_{G_i} associated to each G_i are conjugate to each other, and therefore contain the same numbers of points with (least) period n for every n. Then by Propositions 3.1 and 3.2, the G_i all contain the same (elementary) cycle lengths.

Since all G_i have the same (elementary) cycle lengths, we consider any G_i. By definition of the higher power graph (not given here), the set of (elementary) cycle lengths in G is p-times the set of (elementary) cycle lengths in G_i, thus there exists a cycle in G_i of length k if and only if there exists a cycle in G of length pk. □

Any SFT X can therefore be represented by an associated graph G. Through use of the irreducible components and higher power graph, this G can be reduced to a primitive graph which is far simpler to work with. This will be an advantage in the following results.

4. Results on General Cycle Lengths

4.1. **Primitive Graphs.** The proof of Theorem 3.5, which is equivalent to Theorem 1.1 as noted in Section 3, now follows.

Proof. $\Rightarrow$ Let S be a set closed under $\mathbb{N}$-multiples and cofinite. We will now construct a graph G with the set S as the set of cycle lengths of G. Since S is cofinite, $\exists N \in S$ such that $\forall n \geq N, n \in S$. Build a cycle of length N. $\forall s \in S$ such that $s < N$, build a cycle of length s such that it shares a vertex with the N-cycle, but does not share a vertex with any other cycle of length less than N. This is possible as the N-cycle has N vertices and there exist at most N-1 $s \in S$ such that $s < N$. Call these cycles of length less than N the "small cycles." Then, on the smallest cycle k, build cycles of length $N + i$ such that $i \in 1, ..., k-1$ each sharing a unique vertex with the k-cycle. Call these cycles of length greater than N the "large cycles." Call the resulting graph G.

Every cycle shares a vertex with either the N-cycle or the k-cycle, the N-cycle and the k-cycle themselves sharing a vertex. Thus, there exists a path between vertices in any two cycles, and so G is irreducible. The gcd of the cycles of G is 1 since G contains cycles of length N and $N + 1$. Thus G is aperiodic. Therefore, G is primitive.

Let P be the set of cycle lengths of G. Then let s be any element of S. By construction, $\forall s \in S$ such that $s \leq N$, $\exists$ cycles of length s. Thus if $s \leq N$, the s-cycle already exists within G by construction. Else, $s > N$. Then $\exists i$ such that $0 \leq i \leq k-1$ such that $s \equiv N+i \pmod{k}$ since the $N + i$ cover all k residue classes. Thus $s = N + i + mk$. If $m < 0$, then $s < N + i$ and because $i \in 1, ..., k-1, s < N$. This violates the assumption of $s > N$, hence $m \geq 0$. Thus if $s > N$, the s-cycle can be achieved by going around the $N + i$ cycle once and the k-cycle m times. Therefore, $s \in P$ and since $s \in S$ was arbitrary, $S \subseteq P$.

Let c be any element of P, where C is an associated cycle of G with length c.
There are two cases we consider:

1. C contains at least one edge from a large cycle.
Then by construction C must contain the entire large cycle. If C contains even one large cycle, $c \geq N$ and thus $C \in S$ as S is cofinite.

2. C contains no edges from any large cycle.

Then C must be made up entirely of small cycles. By construction no small cycles share a vertex, thus C is a cycle of length $c = ms$, where $m \in \mathbb{N}$ and $s \in S$. Hence $c \in S$ as S is closed under $\mathbb{N}$-multiples.
Thus, since C was arbitrary, $P \subseteq S$, and therefore S represents the set of cycles of G.

$\Leftarrow$ Let G be a primitive graph and T be the set of cycle lengths of G. $\forall n \in \mathbb{N}$, if $\exists$ a cycle in G of length $z \in \mathbb{N}$, $\exists$ a cycle of length nz obtained by going around the cycle of length z n-times. Hence, T is closed under $\mathbb{N}$-multiples. As well, G being primitive implies it is both irreducible and aperiodic. Thus, by Theorem 4.5.8 and Proposition 2.2.12 from [5], if G is primitive, then T is cofinite. □

4.2. **Irreducible Graphs.** The proof of Theorem 3.6 now follows.

Proof. $\Rightarrow$ Let $R = p \cdot S$, where $p \in \mathbb{N}$ and S is a set closed under $\mathbb{N}$-multiples and cofinite. By Theorem 1, there exists a primitive graph G such that the set of cycle lengths of G is S. Take this graph G and for every directed edge between two vertices I and J in G, create a path of p directed edges and $p-1$ vertices beginning at I and ending at J; all such sets of newly created vertices are disjoint. Call the new graph G_p.

Each cycle length of G has been multiplied by p in G_p, thus $\text{per}(G_p) = p$ since $\text{per}(G)$ $= 1$ as G is primitive. Then take I and J, two vertices of G_p. There are three cases:

1. Both exist in G.
Then there exists a path in G starting at I and terminating at J. Such a path then also exists in G_p, but its length has been multiplied by a factor of p.

2. One vertex exists in G.
Assume I exists in G and J exists only in G_p. By construction, there exist $p-1$ directed edges forming a path starting at a vertex V existing in G and terminating at J. By the previous case there exists a path in G_p starting in I and terminating in V. Then V was at most $p-1$ directed edges away from J, thus there is a path from I to J consisting of the path I to V then V to J.
Assume I exists only in G_p and J exists in G. The proof is similar.

3. Both exist only in G_p.
Then by construction there exist at most $p-1$ directed edges forming a path starting at I and terminating at a vertex existing in G, call it A, and there exist at most $p-1$ directed edges forming a path starting at a vertex existing in G, call it B, and terminating at J. By case 1 there exists a path starting at A and terminating at B. Then there are at most $p-1$ edges I to A and $p-1$ edges B to J, thus there exists a path from I to J consisting of the combination of paths I to A, A to B, B to J.

In each case, for any two vertices I and J there exists a path in G_p starting at I and terminating at J. Therefore G_p is irreducible.

$\Leftarrow$ Let H be an irreducible graph with period p. By Proposition 3.12, H can be associated to a primitive graph G with set of cycle lengths T so that the set of cycle lengths of H is $p \cdot T$ where, by Theorem 3.5, T is a set closed under $\mathbb{N}$-multiples and cofinite. □

4.3. **Arbitrary Graphs.** The proof of Theorem 3.7 now follows.

Proof. $\Rightarrow$ Let $Q = \bigcup_{i=1}^{n} R_i$ where $R_i = p_i \cdot S_i$, where $p_i \in \mathbb{N}$ and S_i is closed under $\mathbb{N}$-multiples and cofinite. By Theorem 3.6, for each $i \in 1, ..., n$, there exists an irreducible graph G_i such that each G_i has set of cycle lengths R_i. Place them together, with no edges connecting any G_i to any other, and call the resulting graph G. As there do not exist any edges connecting vertices from different G_i, the cycles of G are only the cycles of the individual G_i, thus the set of cycles of G is $Q = \bigcup_{i=1}^{n} R_i$.

$\Leftarrow$ Let F be an arbitrary graph. By Proposition 3.11, F can be broken down into irreducible subgraphs F_i for $1 \leq i \leq n$. By Theorem 3.6, the set of cycle lengths of each F_i can be written as $R_i = p_i \cdot T_i$, where T_i is a set that is closed under $\mathbb{N}$-multiples and cofinite and p_i is the period of F_i. Hence the set of cycle lengths of $F = \bigcup_{i=1}^{n} R_i$. □

5. Results on Elementary Cycle Lengths

5.1. **Primitive Graphs.** The proof of Theorem 3.8 now follows.

Proof. $\Rightarrow$ Let S' be either $\{1\}$ or cofinite. If S' is $\{1\}$, then we can create the primitive graph consisting of a single vertex with a self-loop; this graph clearly has only one elementary cycle, with length 1. We then assume S' is a cofinite set. By the same construction found in the proof of Theorem 3.5, use S' to construct a primitive graph G'. Let P' be the set of elementary cycle lengths of G'. The reader can note, since $S \subset P$ in Theorem 3.5 and all cycles created were elementary cycles, $S' \subset P'$.

Then let c' be an element of P', where C' is an elementary cycle of G' of length c'. The two cases seen in the proof of Theorem 3.5 hold, except for a nuance of case 2. Here - the case where C' contains no edges from any large cycle - we need only consider $m = 1$, else C' is not elementary. Since this was the only instance where the fact that S was closed under $\mathbb{N}$-multiples (something not necessarily true of S') was used, $P' \subseteq S'$.

$\Leftarrow$ Let G be a primitive graph. By definition, as G is primitive, G is irreducible and aperiodic. By Lemma 4.5.6 from [5], since G is irreducible, for an arbitrary $v \in V(G)$, the gcd of all lengths of cycles starting and ending at v is $per(G)$. Since $per(G) = 1$, there exist cycles $D_1, D_2, ..., D_l$ beginning and ending at v such that $\gcd(|D_j|) = 1$.

Without loss of generality, we can first remove duplicate cycles. Removing duplicate cycles does not change $\gcd(|D_j|) = 1$, as no cycle length was removed from the gcd calculation. Should a cycle D_i visit v more than only at the beginning and end, it can be written as a concatenation of cycles $C_i^1, C_i^2, ..., C_i^m$, each of which visits v only at the beginning and end. Then, $|D_i| = |C_i^1| + ... + |C_i^m|$, and we claim that we maintain $\gcd(|C_i^1|, ..., |C_i^m|, |D_1|, ..., |D_{i-1}|, |D_{i+1}|, ..., |D_l|) = 1$. To see this, assume for a contradiction $\gcd(|C_i^1|, ..., |C_i^m|, |D_1|, ..., |D_{i-1}|, |D_{i+1}|, ..., |D_l|) \neq 1$. Then $\exists$ a factor q such that $\gcd(|C_i^1|, ..., |C_i^m|, |D_1|, ..., |D_{i-1}|, |D_{i+1}|, ..., |D_l|) = q$. Since $|C_i^1| + ... + |C_i^m| = |D_i|$ and q is a factor of all the $|C_i^j|$, q is a factor of $|D_i|$. Thus $\gcd(|D_j|) = q$, but this is a contradiction since $\gcd(|D_j|) = 1$. We have shown that we can reduce the $D_1, ..., D_l$ to distinct cycles $C_1, ..., C_k$ which only contain v at their beginning and end.

We then break into two cases: $k = 1$ or $k > 1$. If $k = 1$, then $|D_1| = 1$, indicating a self-loop exists at v. Assume for a contradiction that G contains a different elementary cycle C. Then by irreducibility of G, a cycle exists that begins at v and traverses C before returning to v; this path is not just a repeated traversal of the self-loop since we assumed C non-elementary. Then there exists a subcycle C' which contains v only at the beginning and end which is not the self-loop, contradicting $k = 1$. Thus, in this case the only elementary cycle in G is the self-loop at v, and therefore the set of elementary cycle lengths is $\{1\}$.

For the remaining case, assume $k = 1$, and let $S_1 = \{n_1|C_1| + ... + n_k|C_k| : n_i \geq 0\}$. It is well-known that S_1 is cofinite in $\mathbb{N}$ and so $\exists N$ such that $\forall n \geq N$, $n \in S_1$. Let $S_2 = \{n_1|C_1| + ... + n_k|C_k| : \exists j, j' \text{ st } n_j, n_{j'} > 0\}$; we will show that S_2 is also cofinite. Choose an n bigger than N and all possible $|C_i||C_j|$ for $1 \leq i, j \leq k$. Then $\exists n_i \geq 0$ such that $n = n_1|C_1| + ... + n_k|C_k|$. We then break into subcases:
1. If at least 2 of the $n_i > 0$, $n \in S_2$ by definition.
2. Else, since $k > 1$, $\exists i$ such that $\forall j \neq i$, $n_j = 0$. Then choose any of the $j \neq i$. $n = |C_j||C_i| + |C_i|(n_i - |C_j|)$. Recall $n > |C_i||C_j|$. As $n = n_i|C_i|$, this implies $n_i > |C_j|$. Thus $n_i - |C_j| > 0$. Then $n_j = |C_i|$ and $n \in S_2$.
In either case, $n \in S_2$, therefore S_2 is cofinite.

For all $n \in S_2$, $n = n_1|C_1| + ... + n_k|C_k|$, we now construct a cycle of length n by beginning at v and following C_1 n_1-many times, then C_2 n_2-many times ... and finally C_k n_k-many times. Call this cycle C. We can then write $C = C_{i_1}^{t_1} C_{i_2}^{t_2} ... C_{i_\ell}^{t_\ell}$ where $\ell > 1$ and all $t_i > 0$.

We claim C is elementary. If C is not an elementary cycle, there exists C' starting and ending at v such that $C = (C')^p$ for some $p > 1$. Then $C_{i_1}^{t_1} C_{i_2}^{t_2} ... C_{i_\ell}^{t_\ell} = (C')^p$. As each C_{i_j} only visits v at the beginning and end of the cycle it must be that either $C' = C_{i_1}^m$ or $C' = C_{i_1}^{t_1} ... C_{i_j}^{t_j} C_{i_{j+1}}^m$, where $j \geq 1$ and $0 \leq m < t_{j+1}$. In the first case, C' consists of m-many C_{i_1}, thus $(C')^p = C_{i_1}^{mp}$. However, we know C_{i_2} is the next cycle to be followed in order to return to v after $C_{i_1}^{t_1}$. Then $C_{i_1} = C_{i_2}$, however, the C_{i_j} are distinct, thus this is a contradiction. In the second case, $(C')^p = (C_{i_1}^{t_1} ... C_{i_j}^{t_i} C_{i_j+1}^m)^p$, though again we know the after C', $C_{i_{j+1}}$ is the next cycle to be followed before returning to v. Thus $C_{i_1} = C_{i_j+1}$ for some $j \geq 1$. Again, this is a contradiction as the C_{i_j} are distinct. Hence, C is an elementary cycle. Therefore, given a primitive graph G, the set of elementary cycle lengths of G is cofinite. □

5.2. **Irreducible Graphs.** The proof of Theorem 3.9 now follows.

Proof. $\Rightarrow$ Let $R = p \cdot S$ where $p \in \mathbb{N}$ and S is either $\{1\}$ or a cofinite set. In both cases, we can use Theorem 3.8 and the same construction as in the proof of Theorem 3.6 to construct an irreducible graph H whose set of elementary cycle lengths is R.

$\Leftarrow$ Let G be an irreducible graph where R is the set of elementary cycle lengths of G. Let $p = per(G)$. By Proposition 3.12, G has an associated primitive graph G' where the set of elementary cycle lengths of $G = p \cdot S$, S being the set of elementary cycle lengths of G'. By Theorem 3.8, S is $\{1\}$ or cofinite. Since $R = p \cdot S$, either $R = \{p\}$ or $R = p \cdot S$ for S cofinite. □

5.3. **Arbitrary Graphs.** The proof of Theorem 3.10 now follows.

Proof. $\Rightarrow$ Consider any set Q which can be written as $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$ for some finite set U, $p_i \in \mathbb{N}$, and cofinite sets S_i. Then Q can be written as a finite union of singletons and sets of the form $p_i \cdot S_i$, each of which is the set of elementary cycle lengths of an irreducible graph by Theorem 3.9. We can use the same construction as in the proof of Theorem 3.7 to construct a graph F whose set of elementary cycle lengths is Q.

$\Leftarrow$ Let F be an arbitrary graph. By Proposition 3.11, F can be broken down into irreducible subgraphs F_i. By Theorem 3.9, the set of elementary cycle lengths of each F_i can be written as $p_i \cdot T_i$, where T_i is a cofinite set or a singleton and $p_i = per(F_i)$. Hence the set of elementary cycle lengths of F is the union $U \cup \bigcup_{i=1}^{n} p_i \cdot S_i$, where U is the finite union of the singletons and $\{S_i\}_{i=1}^{n}$ is the collection of all T_i which are cofinite. □

References

[1] R. Bowen and O. E. Lanford, III. Zeta functions of restrictions of the shift transformation. In Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), pages 43–49. Amer. Math. Soc., Providence, R.I., 1970.

[2] Emmanuel Jeandel and Pascal Vanier. Periodicity in tilings. In Developments in language theory, volume 6224 of Lecture Notes in Comput. Sci., pages 243–254. Springer, Berlin, 2010.

[3] Ki Hang Kim, Nicholas S. Ormes, and Fred W. Roush. The spectra of nonnegative integer matrices via formal power series. J. Amer. Math. Soc., 13(4):773–806 (electronic), 2000.

[4] D. A. Lind. Entropies and factorizations of topological Markov shifts. Bull. Amer. Math. Soc. (N.S.), 9(2):219–222, 1983.

[5] Douglas Lind and Brian Marcus. An introduction to symbolic dynamics and coding. Cambridge University Press, Cambridge, 1995.

[6] O. M. Šarkovsʹkiĭ. Co-existence of cycles of a continuous mapping of the line into itself. Ukrain. Mat. Ž., 16:61–71, 1964.

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