Jónsson Filters

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Definition

We say that an infinite cardinal $\lambda$ is Jónsson if, for every coloring $F : [\lambda]^\omega \to \lambda$ of the finite subsets of $\lambda$ in $\lambda$-many colors, there is a set $A \in [\lambda]^{\lambda}$ such that $\text{ran}(F \upharpoonright [A]^\omega) \subsetneq \lambda$. 

Lemma (Folklore)

For a cardinal $\lambda$, the following are equivalent:

1. $\lambda$ is Jónsson.
2. For every regular $\chi > \lambda$ and $x \in H(\chi)$, there is a $M \prec (H(\chi), \in, \in, \omega)$ such that $1 | M \cap \lambda | = \lambda$.
3. $\lambda, x \in M$.
4. $\lambda \not\subseteq M$. 

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**Definition**

Suppose that $\lambda$ is a Jónsson cardinal, $\chi > \lambda$ is regular, and $x \in H(\chi)$. We say that $M$ is an $x$-Jónsson model if the following hold:

1. $M \prec ((H(\chi), \in, <_\chi))$
2. $|M \cap \lambda| = \lambda$.
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Suppose $M$ is a Jónsson model. We define the trace of $M$ to be the set $\text{Tr}(M) = M \cap \lambda$. 
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$$\text{Tr}(M) = M \cap \lambda$$
Definition

Suppose that $\lambda$ is a Jónsson cardinal and that $F$ is a uniform filter on $\lambda$. Then we say that $F$ is an $x$-Jónsson filter if there is a regular $\chi > \lambda$ and $x \in H(\chi)$ such that, for any $x$-Jónsson models $M$, we have that $\text{Tr}(M) \in F$. We say that an ideal $I$ is an $x$-Jónsson ideal if its dual $I^*$ is an $x$-Jónsson filter.

Lemma

Suppose $\lambda$ is Jónsson. Given a uniform filter $F$ on $\lambda$, $F$ is a Jónsson filter if and only if there is a coloring $G : [\lambda]^{<\omega} \to \lambda$ such that, for every $A \in [\lambda]^{<\omega}$, we have that $\text{ran}(G | A) \subseteq F$. We say that an ideal $I$ is an $x$-Jónsson ideal if its dual $I^*$ is an $x$-Jónsson filter.
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Definition

Let $\lambda$ be a cardinal of uncountable cofinality, and $S \subseteq \lambda$ be stationary. We say that $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is an $S$-club system if each $C_\delta \subseteq \delta$ is club in $\delta$. 

Suppose we are in the above situation and we have $\bar{I} = \langle I_\delta : \delta \in S \rangle$ is a sequence where each $I_\delta$ is an ideal on $C_\delta$. We can then ask that the pair $(\bar{C}, \bar{I})$ guess clubs in the following sense: Given any club $E \subseteq \lambda$, there are stationary many $\delta \in S$ such that $\text{nacc}(C_\delta) \cap E \notin I_\delta$. 

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**Definition**

Suppose that $\mu$ is a singular cardinal and $S \subseteq \mu^+$ is stationary. Given an $S$-club-system $\bar{C} = \langle C_\delta : \delta \in S \rangle$, for each $\delta \in S$ let $J_\delta$ be the ideal generated by:

1. $\text{acc}(C_\delta)$
2. $\{ \alpha \in C_\delta : \alpha < \beta \}$ for every $\beta < \delta$
3. $\{ \alpha \in C_\delta : \text{cf}(\alpha) < \gamma \}$ for every $\gamma < \mu$
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If each $\delta \in S$ has countable cofinality, and $C_\delta = \{ \delta_n : n < \omega \}$ is a cofinal $\omega$-sequence with $\text{cf}(\delta_{n+1}) > \text{cf}(\delta_n)$, then $J_\delta$ is just the ideal of bounded sets.
Suppose that $S \subseteq \lambda$ is stationary and that $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is an $S$-club system, and $\bar{I} = \langle I_\delta : \delta \in S \rangle$ is a sequence of ideal. We define the ideal $id_p(\bar{C}, \bar{I})$ as follows:

$$A \in id_p(\bar{C}, \bar{I}) \text{ if and only if there is a club } E \subseteq \lambda \text{ such that the set } \{ \delta \in S : E \cap A \cap nacc(C_\delta) \notin I_\delta \} \text{ is non-stationary.}$$
Suppose that $S \subseteq \lambda$ is stationary and that $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is an $S$-club system, and $\bar{I} = \langle I_\delta : \delta \in S \rangle$ is a sequence of ideal. We define the ideal $id_p(\bar{C}, \bar{I})$ as follows:

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In particular, $id_p(\bar{C}, \bar{I})$ is proper precisely when the pair $(\bar{C}, \bar{I})$ guesses clubs in the sense we described earlier.
Theorem (Shelah)

Given a singular cardinal $\mu$, we can find:

1. A stationary $S \subseteq \mu^+ \setminus \mu$, and
2. an $S$-club system $\vec{C} = \langle C_\delta : \delta \in S \rangle$

such that $\text{id}_p(\vec{C}, \vec{I})$ is proper, where $\vec{I} = \langle J_\delta : \delta \in S \rangle$. 

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Theorem (Shelah)

Suppose that $\mu$ is a singular cardinal with $\mu^+$ Jónsson. If $S$, $\bar{I}$, and $\bar{C}$ are as above, then there is a coloring $F : [\lambda]^{<\omega} \to \lambda$ such that, for any $A \in [\lambda]^{\lambda}$,

$$\lambda \setminus \text{ran}(F \upharpoonright [A]^{<\omega}) \in id_p(\bar{C}, \bar{I}).$$
Definition
Let $I$ an ideal over a set $A$, and $\theta$ a cardinal. We say that $I$ is weakly $\theta$-saturated if any collection of sets partitioning $A$ into disjoint $I$-positive sets has size $< \theta$.

Definition
Let $I$ be an ideal over a set $A$, and let $\kappa$ be a regular cardinal. We say that $I$ is $\kappa$-indecomposable if $I$ is closed under increasing unions of length $\kappa$. 
Lemma

Suppose that $\lambda$ is a Jónsson cardinal. If there is a Jónsson ideal $I$ on $\lambda$, and $I$ is not weakly $\theta$-saturated for some $\theta \leq \lambda$, then there is a coloring $c : [\lambda]^{<\omega} \to \theta$ such that $\text{ran}(C \upharpoonright [A]^{<\omega}) = \theta$. 

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Lemma

Suppose that $\lambda$ is a Jónsson cardinal. If there is a Jónsson ideal $I$ on $\lambda$, and $I$ is not weakly $\theta$-saturated for some $\theta \leq \lambda$, then there is a coloring $c : [\lambda]^{<\omega} \to \theta$ such that $\text{ran}(C \upharpoonright [A]^{<\omega}) = \theta$.

Let $\langle A_i : i < \theta \rangle$ be a partition of $\lambda$ into $\theta$-many $I$ positive sets. Let $F$ witness that $I$ is a Jónsson ideal, and define $c : [\lambda]^{<\omega} \to \theta$ by

$$c(s) = i \text{ if } F(s) \text{ lies in } A_i.$$ 

For any $A \in [\lambda]^\lambda$, we then have $\text{ran}(c \upharpoonright [A]^{<\omega}) = \theta$. 
Lemma

Suppose that $\lambda$ is a regular Jónsson cardinal, and that $I$ is a Jónsson ideal over $\lambda$.

- If $\lambda$ is inaccessible, then $I$ is $\theta$-indecomposable for unboundedly-many regular $\theta < \lambda$.
- If $\lambda = \mu^+$ for $\mu$ singular, then $I$ is $\theta$-indecomposable for unboundedly-many regular $\theta < \mu$.
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As a corollary, if $\lambda$ is Jónsson and $I$ is a Jónsson filter on $\lambda$, then there is some regular $\theta < \lambda$ such that $I$ is weakly $\theta$-saturated and $\theta$-indecomposable.
Suppose that $\lambda$ is an ordinal of uncountable cofinality.

- We say that $S$ reflects at an ordinal $\alpha < \lambda$ of uncountable cofinality if $S \cap \alpha$ is stationary in $\alpha$.
- Given a collection $\mathcal{S}$ of stationary subsets of $\lambda$, we say that $\mathcal{S}$ reflects simultaneously if there is some ordinal $\alpha < \lambda$ of uncountable cofinality such that each $S \in \mathcal{S}$ reflects at $\alpha$. 
If $I$ is an ideal on $\lambda$, denote:

1. $\text{Comp}(I)$ is the largest cardinal $\theta$ such that $I$ is $\theta$-complete.
2. $\text{Wsat}(I)$ is the least cardinal $\theta$ such that $I$ is weakly $\theta$-saturated.
3. $\text{Indec}(I) = \{\theta < \lambda : I \text{ is } \theta\text{-indecomposable}\}$.
4. $S(I) = \{\alpha < \lambda : \text{Wsat}(I) \leq \text{cf}(\alpha) < \alpha \text{ and } \text{cf}(\alpha) \in \text{Indec}(I)\}$. 
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**Theorem (Eisworth)**

Let $I$ be an ideal on a cardinal $\lambda$. If there is a cardinal $\theta$ such that $I$ is weakly $\theta$-saturated and $\theta$-indecomposable, then

1. $S(I)$ is stationary, and
2. Any fewer than $\text{Comp}(I)$-many stationary subsets of $S(I)$ reflect simultaneously
Definition

Suppose that $\delta$ is an ordinal of uncountable cofinality, and $S \subseteq \delta$. We define $rk(S)$ as follows:

1. $rk(S) = 0$ if $S$ is non-stationary.
2. $rk(S) \geq 1$ if and only if $S$ is stationary.
3. For $i \geq 1$, we say $rk(S) \geq i + 1$ if and only if $\{\alpha < \delta : rk(S \cap \alpha) \geq i\}$ is stationary.
4. If $i$ is a limit ordinal, then $rk(S) \geq i$ if and only if $rk(S) \geq j$ for every $j < i$.

If we think of simultaneous reflection as a notion of width for stationary reflection, then the rank of a stationary set is analogous to its depth of stationary reflection.
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If we think of simultaneous reflection as a notion of width for stationary reflection, then the rank of a stationary set is analogous to its depth of stationary reflection.
Proposition

Suppose that $\lambda$ is an inaccessible Jónsson cardinal. If there is a $\theta$-complete Jónsson filter on $\lambda$, then $\text{rk}(S) \geq \theta$ for every stationary $S \subseteq \lambda$. 
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Suppose that $\lambda$ is an inaccessible Jónsson cardinal. If there is a $\theta$-complete Jónsson filter on $\lambda$, then $\text{rk}(S) \geq \theta$ for every stationary $S \subseteq \lambda$.

A corollary to Lemma 1.9 from Chapter III of Cardinal Arithmetic is that, if $\lambda$ is an inaccessible Jónsson cardinal and not $\omega$-Mahlo, then we can build a Jónsson filter on $\lambda$. The above theorem then furnishes an easy proof of the fact that any inaccessible Jónsson cardinal must be at least $\omega$-Mahlo.
Let $x \in H(\chi)$ be a parameter witnessing the existence of an $x$-Jónsson Filter $F$. Fix a stationary $S \subseteq \lambda$, and a $\lim(\lambda)$-club system $\langle e_\delta : \delta \in \lim(\lambda) \rangle$ with the property that

- If $\text{cf}(\delta) > \omega$ and $S \cap \delta$ is stationary, then for every $\alpha \in e_\delta$, we have $\text{rk}(S \cap \alpha) < \text{rk}(S \cap \delta)$. Otherwise, we demand that $S \cap \alpha$ is non-stationary.
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By replacing $S$ with $S \cap C$ for an appropriate club $C \subseteq \lambda$, we may assume that $\text{rk}(S \cap \delta) < \text{rk}(S)$ for every $\delta < \lambda$. We build a tower of Jónsson models of height $\theta$ as follows:

1. First, we build Jónsson models $N_\xi$ for $\xi < \theta$ such that $\theta + 1 \subseteq N_\xi$ and $N_\xi \prec (H(\chi), \in, <_\chi, x, \bar{e}, \langle N_\epsilon : \epsilon < \xi \rangle)$.
2. We then inductively define $M_\xi$ for $\xi \leq \theta$ by setting $M_\xi = \bigcap_{\epsilon < \xi} N_\epsilon$ for $1 \leq \xi \leq \theta$. 
We can find some $\delta \in S$ such that, letting $\beta_\xi = \min(M_\xi \cap \lambda \setminus \delta)$, we have:

1. Each $\beta_\xi$ is a limit ordinal of uncountable cofinality, and $\langle \beta_\xi : 1 \leq \xi \leq \theta \rangle$ forms a decreasing sequence of ordinals above $\delta$.
2. $\delta \in e_{\beta_\xi}$ for each $\xi$.
3. If $\epsilon < \xi$, then $\beta_\epsilon \in e_{\beta_\xi}$.
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We then argue inductively that $rk(S \cap \beta_\xi) \geq \xi$ for each $\xi$.

1. For $\xi = 1$, we can show that $\delta \in S \cap e_{\beta_1}$ gives us that every club $d \subseteq \beta_1$ with $d \in N'_1$ contains $\delta$ as an accumulation point. So by elementarity, $S \cap \beta_1$ is stationary.
2. For $\xi > 1$, we have that $rk(S \cap \beta_\epsilon) \geq \epsilon$ for each $\epsilon < \xi$. As $\epsilon \in e_{\beta_\xi}$ it follows from our definition of $e_{\beta_\xi}$ that $rk(S \cap \beta_\xi) > \epsilon$ for each $\epsilon < \xi$. 
Thank you for your time.