Universal automorphisms of $\mathcal{P}(\omega)/\text{fin}$

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\[ \mathcal{P}(\omega)/\text{fin} \text{ and its trivial self-maps} \]

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- Every function \(f : \omega \to \omega\) induces a function \(f^\uparrow : \mathcal{P}(\omega)/\text{fin} \to \mathcal{P}(\omega)/\text{fin}\).
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- Every function \( f : \omega \to \omega \) induces a function \( f^\uparrow : \mathcal{P}(\omega)/\text{fin} \to \mathcal{P}(\omega)/\text{fin} \).
- If \( f \) is a mod-finite permutation of \( \omega \), then the map \( f^\uparrow \) induced by \( f \) is an automorphism of \( \mathcal{P}(\omega)/\text{fin} \).
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$\mathcal{P}(\omega)/\text{fin}$ and its trivial self-maps

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- If $f$ is a mod-finite permutation of $\omega$, then the map $f^{\uparrow}$ induced by $f$ is an automorphism of $\mathcal{P}(\omega)/\text{fin}$. Automorphisms of this kind are called trivial.

A “mod-finite permutation” of $\omega$ means a bijection $A \to B$, where both $A$ and $B$ are co-finite subsets of $\omega$. 

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**Theorem (Parovičenko, 1963)**

*Every Boolean algebra of size \( \leq \aleph_1 \) embeds in \( \mathcal{P}(\omega)/\text{fin} \).*
The *successor map* on $s : \omega \to \omega$ is an example of a mod-finite permutation:

\[
\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots
\]

Its lifting to $\mathcal{P}(\omega)/\text{fin}$, namely

\[
s^\uparrow([A]) = [A + 1]
\]

is called the *shift map*. 
Whether every automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial is independent of ZFC:

The Continuum Hypothesis implies there are $2^{\aleph_1}$ automorphisms of $\mathcal{P}(\omega)/\text{fin}$. The number of trivial automorphisms is only $2^{\aleph_0}$, so CH implies that "most" automorphisms are nontrivial.

On the other hand, Shelah proved it is consistent with ZFC that every automorphism of $\mathcal{P}(\omega)/\text{fin}$ is trivial. Shelah and Steprāns later showed that this is a consequence of PFA, and Veličković ultimately weakened the assumption to OCA+MA.
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non-trivial autohomeomorphisms

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Mapping one automorphism into another

Suppose \( A \) and \( B \) are Boolean algebras, and that \( \alpha \) and \( \beta \) are automorphisms of \( A \) and \( B \), respectively.
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We say that $\alpha$ embeds in $\beta$, and we write $\alpha \hookrightarrow \beta$, if there is an embedding $e : A \to B$ such that $e \circ \alpha = \beta \circ e$. 

Equivalently, $\alpha \hookrightarrow \beta$ if there is a subalgebra $C$ of $A$ such that $(B, \beta)$ is isomorphic to $(C, \alpha \restriction C)$. 

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\[
\begin{array}{ccc}
\mathbb{B} & \xrightarrow{\beta} & \mathbb{B} \\
\downarrow{e} & & \downarrow{e} \\
\mathbb{A} & \xrightarrow{\alpha} & \mathbb{A}
\end{array}
\]
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The main result of this talk is an analogue for automorphisms of Parovičenko’s result for algebras:

**Main Theorem**

Let $f$ be a mod-finite permutation of $\omega$. If $A$ is a Boolean algebra of size $\leq \aleph_1$ and $\alpha : A \to A$ is an automorphism, then following are equivalent:

1. $\alpha \hookrightarrow f^\uparrow$.
2. $\alpha \upharpoonright C \hookrightarrow f^\uparrow$ for every countable, $\alpha$-invariant subalgebra $C$ of $A$.
3. there is no “finite obstruction” to embedding $\alpha$ in $f^\uparrow$. 
an example of a finite obstruction

Recall that $s$ denotes the successor map $n \mapsto n + 1$. 
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**Proposition**

If $x \in \mathcal{P}(\omega)/\text{fin}$ with $[\emptyset] \neq x \neq [\omega]$, then $s^\uparrow(x) \not\leq x$. 

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Universal automorphisms of $\mathcal{P}(\omega)/\text{fin}$
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So, for example, if $\alpha : A \to A$ has nontrivial fixed points, then $\alpha$ does not embed in $s^\uparrow$, because this proposition provides an obstruction.
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**Proposition**

If $x \in \mathcal{P}(\omega)/\text{fin}$ with $[\emptyset] \neq x \neq [\omega]$, then $s^\uparrow(x) \not\leq x$.

So, for example, if $\alpha : \mathbb{A} \to \mathbb{A}$ has nontrivial fixed points, then $\alpha$ does not embed in $s^\uparrow$, because this proposition provides an obstruction. In fact, one may show that this proposition provides the only possible finite obstruction to embedding in the shift map:

**Theorem**

Let $\alpha$ be an automorphism of a Boolean algebra $\mathbb{A}$ with $|\mathbb{A}| \leq \aleph_1$. Then $\alpha$ embeds in the shift map $s^\uparrow$ if and only if $\alpha(x) \not\leq x$ whenever $0 \neq x \neq 1$. 
Let $t$ denote a permutation of $\omega$ that consists of infinitely many $\mathbb{Z}$-like orbits:

\[ \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]

\[ t \]

\[ \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]

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t \\
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\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \\
\vdots \quad \vdots \quad \vdots
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**Theorem**

*There are no finite obstructions to embedding in $t^\uparrow$. In fact, every automorphism of every countable Boolean algebra embeds in $t^\uparrow$.***
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$t$

\[ \cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \]

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\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]

**Theorem**

*There are no finite obstructions to embedding in $t^\uparrow$. In fact, every automorphism of every countable Boolean algebra embeds in $t^\uparrow$. Consequently (applying the “main theorem”), every automorphism of a Boolean algebra of size $\leq \aleph_1$ embeds in $t^\uparrow$.***
Corollary

Assuming the Continuum Hypothesis, every automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in $t^\uparrow$. 

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Corollary

Assuming the Continuum Hypothesis, every automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in $t^\uparrow$.

Let us say that an automorphism of $\mathcal{P}(\omega)/\text{fin}$ is universal if every other automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in it. Thus, according to the corollary above, CH implies there is a universal automorphism of $\mathcal{P}(\omega)/\text{fin}$. 
a corollary

**Corollary**

*Assuming the Continuum Hypothesis, every automorphism of \( P(\omega)/\text{fin} \) embeds in \( t^\uparrow \).*

Let us say that an automorphism of \( P(\omega)/\text{fin} \) is *universal* if every other automorphism of \( P(\omega)/\text{fin} \) embeds in it. Thus, according to the corollary above, CH implies there is a universal automorphism of \( P(\omega)/\text{fin} \).

**Theorem**

*Assuming the Continuum Hypothesis, \( t^\uparrow \) embeds in \( 2^{\aleph_1} \) distinct automorphisms of \( P(\omega)/\text{fin} \). Because a composition of embeddings is an embedding, CH implies that there are \( 2^{\aleph_1} \) distinct universal automorphisms of \( P(\omega)/\text{fin} \).*
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a sketch of the proof: what won’t work

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Perhaps the most obvious strategy for proving the universality of $t^\uparrow$ is as follows:

• Begin with an automorphism $\alpha : A \rightarrow A$ where $|A| \leq \aleph_1$. Write $A$ as an increasing union of countable, $\alpha$-invariant subalgebras $\bigcup_{\xi<\omega_1} A_\xi$ and let $\alpha_\xi = \alpha \upharpoonright A_\xi$ for all $\xi < \omega_1$. 
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Perhaps the most obvious strategy for proving the universality of $t^\uparrow$ is as follows:

- Begin with an automorphism $\alpha : \mathbb{A} \rightarrow \mathbb{A}$ where $|\mathbb{A}| \leq \aleph_1$. Write $\mathbb{A}$ as an increasing union of countable, $\alpha$-invariant subalgebras $\bigcup_{\xi < \omega_1} \mathbb{A}_\xi$ and let $\alpha_\xi = \alpha \upharpoonright \mathbb{A}_\xi$ for all $\xi < \omega_1$.
- We already mentioned that every automorphism of a countable Boolean algebra embeds in $t^\uparrow$, so fix an embedding $e_0 : \mathbb{A}_0 \rightarrow \mathcal{P}(\omega)/\text{fin}$ such that $e_0 \circ \alpha_0 = t^\uparrow \circ e_0$. 

\[ \begin{align*} \alpha & \quad \vdots \quad \mathbb{A} \\ \cup & \quad \mathbb{A}_\xi \\ \vdots & \quad \mathbb{A}_2 \\ \cup & \quad \mathbb{A}_1 \\ \cup & \quad \mathbb{A}_0 \\ \alpha_0 & \quad \overset{e_0}{\longrightarrow} \quad t^\uparrow \end{align*} \]
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- Lift $e_0$ to an embedding $e_1$ of $A_1 \supseteq A_0$ into $\mathcal{P}(\omega)/\text{fin}$ such that $e_1 \circ \alpha_1 = t^\uparrow \circ e_1$. 
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- Lift $e_0$ to an embedding $e_1$ of $A_1 \supseteq A_0$ into $\mathcal{P}(\omega)/\text{fin}$ such that $e_1 \circ \alpha_1 = t^\uparrow \circ e_1$.
- Continue this up through all the $\alpha_\xi$, taking unions at limit stages.
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- Lift $e_0$ to an embedding $e_1$ of $\mathbb{A}_1 \supseteq \mathbb{A}_0$ into $\mathcal{P}(\omega)/\text{fin}$ such that $e_1 \circ \alpha_1 = t^\uparrow \circ e_1$.

- Continue this up through all the $\alpha_\xi$, taking unions at limit stages.

- In the end, $e = \bigcup_{\xi < \omega_1} e_\xi$ embeds $\alpha$ in $t^\uparrow$. 
a sketch of the proof: what won’t work

Perhaps the most obvious strategy for proving the universality of $t^\uparrow$ is as follows:

- Begin with an automorphism $\alpha : A \to A$ where $|A| \leq \aleph_1$. Write $A$ as an increasing union of countable, $\alpha$-invariant subalgebras $\bigcup_{\xi < \omega_1} A_\xi$ and let $\alpha_\xi = \alpha \upharpoonright A_\xi$ for all $\xi < \omega_1$.

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- Lift $e_0$ to an embedding $e_1$ of $A_1 \supset A_0$ into $\mathcal{P}(\omega)/\text{fin}$ such that $e_1 \circ \alpha_1 = t^\uparrow \circ e_1$.

- Continue this up through all the $\alpha_\xi$, taking unions at limit stages.

- In the end, $e = \bigcup_{\xi < \omega_1} e_\xi$ embeds $\alpha$ in $t^\uparrow$. 
a sketch of the proof: how to fix it

This strategy does not work as stated, but it can be made to work by choosing the $A_\xi$ more carefully. Specifically:

- Fix a continuous chain $\langle M_\xi : \xi < \omega_1 \rangle$ of countable elementary submodels of a suitable fragment of the set-theoretic universe.
- For each $\xi < \omega_1$, let $A_\xi = A \cap M_\xi$ and define $\alpha_\xi = \alpha \upharpoonright A_\xi$ as before.
- The elementarity between the models makes $\alpha_\xi$ behave nicely with respect to $\alpha_{\xi+1}$, and makes it possible for any $e_\xi : A_\xi \to \mathcal{P}(\omega)/\text{fin}$ embedding $\alpha_\xi$ into $t^\uparrow$ to be lifted to some $e_{\xi+1} : A_{\xi+1} \to \mathcal{P}(\omega)/\text{fin}$ embedding $\alpha_{\xi+1}$ into $t^\uparrow$.
- Then the argument outlined on the previous slide can succeed.
what if CH fails?

Question

Is $\neg \text{CH}$ consistent with the existence of universal automorphisms of $\mathcal{P}(\omega)/\text{fin}$? Might ZFC even imply the existence of such automorphisms?
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I have no idea. A more tractable question might be:

**Question**

Does OCA + MA imply the existence of universal automorphisms of $\mathcal{P}(\omega)/\text{fin}$?
OCA + MA seems to decide most questions about $\mathcal{P}(\omega)/\text{fin}$, and work of Farah (and others) seems to indicate that it is something of an optimal hypothesis for ensuring $\mathcal{P}(\omega)/\text{fin}$ has as few self-maps as possible.
why OCA + MA?

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- OCA + MA implies that all automorphisms of \( \mathcal{P}(\omega)/\text{fin} \) are trivial.
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- OCA + MA implies that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial.
- While ZFC implies the existence of nontrivial self-embeddings of $\mathcal{P}(\omega)/\text{fin}$ (by recent work of Dow), OCA + MA restricts the form of these embeddings, and ensures they are “close” to trivial.
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- While ZFC implies the existence of nontrivial self-embeddings of $\mathcal{P}(\omega)/\text{fin}$ (by recent work of Dow), OCA + MA restricts the form of these embeddings, and ensures they are “close” to trivial.
- For example, OCA + MA implies that the shift map does not embed into its inverse, and vice versa.
two more permutations of \( \omega \)

Let \( r \) denote a permutation of \( \omega \) that consists of infinitely many finite cycles, one of size \( n! \) for every \( n \):

\[
\begin{array}{c}
\cdots \\
\vdots \\
\end{array}
\]

Let \( t \lor r \) denote the permutation of \( \omega \) obtained by putting a copy of \( t \) next to a copy of \( r \):

\[
\begin{array}{c}
\cdots \\
\vdots \\
\end{array}
\]
Every trivial automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in either $t^\uparrow$ or in $(t \lor r)^\uparrow$.
universal automorphisms of $\mathcal{P}(\omega)/\text{fin}$
universal automorphisms with CH
universal automorphisms without CH

a jointly universal pair

Theorem

*Every trivial automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in either $t^\uparrow$ or in $(t \lor r)^\uparrow$. Hence OCA + MA implies that every automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in either $t^\uparrow$ or in $(t \lor r)^\uparrow$.**
Every trivial automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in either $t^\uparrow$ or in $(t \lor r)^\uparrow$. Hence $\text{OCA} + \text{MA}$ implies that every automorphism of $\mathcal{P}(\omega)/\text{fin}$ embeds in either $t^\uparrow$ or in $(t \lor r)^\uparrow$.

More specifically, if $f$ is a mod-finite permutation of $\omega$, then $f^\uparrow$ embeds in $t^\uparrow$ if it has no “cyclic part” (i.e., if $f$ contains only finitely many finite cycles), and otherwise it embeds in $(t \lor r)^\uparrow$.

$\text{OCA} + \text{MA}$ implies that $r^\uparrow$ does not embed in $t^\uparrow$.

Thus $t^\uparrow$ is not universal under $\text{OCA} + \text{MA}$.
what about \((t \lor r)^\uparrow\)?

\[
\ldots \text{but } (t \lor r)^\uparrow \text{ might be.}
\]
what about $(t ∨ r)^↑$?

. . . but $(t ∨ r)^↑$ might be.

Recall that $ω^*$ denotes the Stone space of $\mathcal{P}(ω)/\text{fin}$, and that $f^*$ denotes the self-homeomorphism of $ω^*$ induced by a mod-finite permutation $f$ of $ω$; i.e., $f^* = \text{Stone}(f^↑)$. 
what about \((t \vee r)^\uparrow\)?

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Recall that \(\omega^*\) denotes the Stone space of \(\mathcal{P}(\omega)/\text{fin}\), and that \(f^*\) denotes the self-homeomorphism of \(\omega^*\) induced by a mod-finite permutation \(f\) of \(\omega\); i.e., \(f^* = \text{Stone}(f^\uparrow)\).

**Theorem**

*Assuming there are no nontrivial automorphisms of \(\mathcal{P}(\omega)/\text{fin}\) (e.g., under OCA + MA), then \((t \vee r)^\uparrow\) is universal if and only if there is a continuous function \(q : \omega^* \to \omega^*\) such that \(q \circ r^* = s^* \circ q\).*
what about \((t \lor r)^\uparrow\)?

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Recall that \(\omega^*\) denotes the Stone space of \(\mathcal{P}(\omega)/\text{fin}\), and that \(f^*\) denotes the self-homeomorphism of \(\omega^*\) induced by a mod-finite permutation \(f\) of \(\omega\); i.e., \(f^* = \text{Stone}(f^\uparrow)\).

**Theorem**

*Assuming there are no nontrivial automorphisms of \(\mathcal{P}(\omega)/\text{fin}\) (e.g., under OCA + MA), then \((t \lor r)^\uparrow\) is universal if and only if there is a continuous function \(q : \omega^* \to \omega^*\) such that \(q \circ r^* = s^* \circ q\).*

OCA + MA implies that any such function \(q\) must be nontrivial; i.e., it cannot be induced by a function \(\omega \to \beta \omega\).
open questions

Question

Is \((t \lor r)^\uparrow\) a universal automorphism of \(P(\omega)/\text{fin}\) under OCA + MA?
open questions

Question

Is \((t \vee r)^\uparrow\) a universal automorphism of \(\mathcal{P}(\omega)/\text{fin}\) under OCA + MA?

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Does every automorphism of \(\mathcal{P}(\omega)/\text{fin}\) embed in a trivial automorphism?
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Is $(t \lor r) \uparrow$ a universal automorphism of $\mathcal{P}(\omega)/\text{fin}$ under OCA + MA?

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Does every automorphism of $\mathcal{P}(\omega)/\text{fin}$ embed in a trivial automorphism?

Question

Does CH imply that the shift map is conjugate/isomorphic to its inverse?
The end

Thank you for listening