Distributivity in residuated structures

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The setting

Definition:

A *residuated binar* is an expansion of a lattice $\mathbf{A}$ by binary operations $\cdot$, $\backslash$, $/$ that satisfy the law of residuation,

\[
\forall x, y, z \in \mathbf{A}, \quad y \leq x \backslash z \iff x \cdot y \leq z \iff x \leq z / y.
\]

Note: Residuated lattices are residuated binars for which $\cdot$ is associative and commutative.
The setting

Definition:
A *residuated binar* is an expansion of a lattice $A$ by binary operations $\cdot, \setminus, /$ that satisfy the law of residuation, i.e., for all $x, y, z \in A$,

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One may think of the law of residuation as a “finite approximation” of an infinite distributive property.
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If \( A = (\land, \lor, \cdot, \backslash, /) \) is a residuated binar, then multiplication preserves existing joins in each argument.
One may think of the law of residuation as a “finite approximation” of an infinite distributive property.

If $\mathbf{A} = (A, \wedge, \vee, \cdot, \setminus, /)$ is a residuated binar, then multiplication preserves existing joins in each argument.

In other words, if $X, Y \subseteq A$ and $\bigvee X$ and $\bigvee Y$ exist, then

$$\bigvee X \cdot \bigvee Y = \bigvee \{xy : x \in X, y \in Y\}.$$
The division operations \ and / also satisfy strong distributive properties.
Residuation and distributivity (cont.)

The division operations \( \setminus \) and \( / \) also satisfy strong distributive properties.

Divisions preserve all existing meets in the numerator, and convert all existing joins in the denominator to meets.
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In other words, if \( X, Y \subseteq A \) and \( \bigvee X, \bigwedge Y \) exist, then for any \( z \in A \) each of \( \bigwedge_{x \in X} x \setminus z, \bigwedge_{x \in X} z / x, \bigwedge_{y \in Y} z \setminus y, \) and \( \bigwedge_{y \in Y} y / z \) exists and

\[
\begin{align*}
z \setminus (\bigwedge Y) &= \bigwedge_{y \in Y} x \setminus y, \\
(\bigwedge Y) / z &= \bigwedge_{y \in Y} y / z.
\end{align*}
\]

\[
\begin{align*}
(\bigvee X) \setminus z &= \bigwedge_{x \in X} x \setminus z, \\
z / (\bigvee X) &= \bigwedge_{x \in X} z / x.
\end{align*}
\]
There is a partial converse to the above.
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**Proposition:**
Let \( A \) be a complete lattice expanded by a binary operation \( \cdot \). Then \( \cdot \) is residuated if it distributes over arbitrary joins in each coordinate.

\[
\begin{align*}
    x \cdot (y \lor z) &= (x \cdot y) \lor (x \cdot z), \\
    (x \lor y) \cdot z &= (x \cdot z) \lor (y \cdot z).
\end{align*}
\]
There is a partial converse to the above.

**Proposition:**

Let $\mathbf{A}$ be a complete lattice expanded by a binary operation $\cdot$. Then $\cdot$ is residuated if it distributes over arbitrary joins in each coordinate.

In particular, if $\cdot$ is a binary operation on a *finite* lattice then $\cdot$ is residuated if it satisfies $x \cdot (y \lor z) = (x \cdot y) \lor (x \cdot z)$ and $(x \lor y) \cdot z = (x \cdot z) \lor (y \cdot z)$. 
In more finitary terms, we have:

\[
\begin{align*}
\text{(·∨)} & \quad x (y ∨ z) = xy ∨ xz, \\
\text{∨·} & \quad (x ∨ y) z = xz ∨ yz, \\
\text{(∧)} & \quad x \ (y ∧ z) = x \ y ∧ x \ z, \\
\text{∧/} & \quad (x ∧ y) / z = x / z ∧ y / z, \\
\text{(∨/)} & \quad x / (y ∨ z) = x / y ∧ x / z, \\
\text{∨\)} & \quad (x ∨ y) \ z = x \ z ∧ y \ z.
\end{align*}
\]
In more finitary terms, we have:

**Proposition:**
Let \( A \) be a residuated binar. Then \( A \) satisfies the following.

\[
\begin{align*}
\cdot \lor & \quad x(y \lor z) = xy \lor xz. \\
\lor \cdot & \quad (x \lor y)z = xz \lor yz. \\
\land \lor & \quad x\lnot(y \land z) = x\lnot y \land x\lnot z. \\
\lor \land & \quad (x \land y)z = x/z \land y/z. \\
\land \lor & \quad x/(y \lor z) = x/y \land x/z. \\
\lor \land & \quad (x \lor y)\lnot z = x\lnot z \land y\lnot z.
\end{align*}
\]
Consider the following nontrivial distributive laws:

\[
\begin{align*}
(x \land y) \land z &= xy \land xz, \\
(x \land y) z &= xz \land yz, \\
(x \lor y) \lor z &= x \lor y \lor z, \\
(x \lor y) \lor z &= x \lor z \lor y, \\
(x \land y) \lor z &= x \land z \lor y, \\
(x \land y) \lor z &= x \land z \lor y.
\end{align*}
\]
Consider the following nontrivial distributive laws:

\[ x(y \land z) = xy \land xz \]  \hspace{1cm} (\cdot \land)

\[ (x \land y)z = xz \land yz \]  \hspace{1cm} (\land \cdot)

\[ x \setminus (y \lor z) = x \setminus y \lor x \setminus z \]  \hspace{1cm} (\setminus \lor)

\[ (x \lor y) / z = x / z \lor y / z \]  \hspace{1cm} (\lor /)

\[ (x \land y) \setminus z = x \setminus z \lor y \setminus z \]  \hspace{1cm} (\land \setminus)

\[ x / (y \land z) = x / y \lor x / z \]  \hspace{1cm} (\lor \land)
Consider the following nontrivial distributive laws:

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x(y \land z) = xy \land xz \\
(x \land y)z = xz \land yz \\
x \setminus (y \lor z) = x \setminus y \lor x \setminus z \\
(x \lor y)/z = x/z \lor y/z \\
(x \land y)\setminus z = x\setminus z \lor y\setminus z \\
x/(y \land z) = x/y \lor x/z
\]

The main purpose of this work is to understand the poset of subvarieties axiomatized by the nontrivial distributive laws.
A residuated binar is *semilinear* if it is a subdirect product of chains. All six nontrivial distributive laws hold in semilinear residuated binars.
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**Proposition (Blount and Tsinakis, 2003):**

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, \setminus, \div, e)$ be a residuated lattice satisfying

$$(x \vee y) \wedge e = (x \wedge e) \vee (y \wedge e).$$

Then

- $e \leq x / y \vee y / x$ iff $(\setminus \wedge)$ iff $(\vee \div)$.
- $e \leq y \setminus x \vee x \setminus y$ iff $(\wedge \setminus)$ iff $(\setminus \vee)$.
In residuated binars, the above fails.
The general picture

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Let $A$ be the residuated binar with lattice reduct $\{\bot, a, b, \top\}$, where $\bot < a, b < \top$, defined in the below.
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Then $A | = (\land \land)$, but $A \neq (\lor \land)$. 

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In residuated binars, the above fails.

Let $A$ be the residuated binar with lattice reduct $\{\bot, a, b, \top\}$, where $\bot < a, b < \top$, defined in the below.

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Then $A \models (\land \land \land)$, but $A \not\models (\land \lor \land \lor)$. 
Nevertheless, there are implications among the nontrivial distributive laws...
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**Proposition (WF, P. Jipsen 2018):**

Let $A$ be a residuated binar whose lattice reduct is distributive. Then if $A$ satisfies both $(\lor/) \land (\land\backslash)$, $A$ also satisfies $(\backslash\lor)$. 
Nevertheless, there are implications among the nontrivial distributive laws...

Proposition (WF, P. Jipsen 2018):

Let $A$ be a residuated binar whose lattice reduct is distributive. Then if $A$ satisfies both $(\lor/)$ and $(\land\backslash)$, $A$ also satisfies $(\backslash\lor)$.

Proof: Note first that we can rewrite the identities $(\land\backslash)$ and $(\backslash\lor)$. 
We have that \((\land\|)\) is equivalent to

\[(x \land y)\|(z \land w) \leq x\|z \lor y\|w,\]

and \((\|\lor)\) is equivalent to

\[(x \lor y)\|(z \lor w) \leq x\|z \lor y\|w.\]
We have that \((\land/)\) is equivalent to

\[(x \land y) \setminus (z \land w) \leq x \setminus z \lor y \setminus w,
\]

and \((/\lor)\) is equivalent to

\[(x \lor y) \setminus (z \lor w) \leq x \setminus z \lor y \setminus w.
\]

Let \(u \leq (x \lor y) \setminus (z \lor w)\). Then by residuation \(x \lor y \leq (z \lor w)/u\), and by \((\lor/)\) we have \(x \lor y \leq z/u \lor w/u\).
This gives $x \lor y = (x \lor y) \land (z/u \lor w/u)$. 

Notice: 
$x_1 \leq z/u = \Rightarrow u \leq x_1 \land z \leq (x_1 \land y_2)$ 
$x_2 \leq w/u = \Rightarrow u \leq x_2 \land w \leq (x_2 \land y_1)$ 
$y_1 \leq z/u = \Rightarrow u \leq y_1 \land z \leq (x_2 \land y_1)$ 
$y_2 \leq w/u = \Rightarrow u \leq y_2 \land w \leq (x_1 \land y_2)$. 

The general picture (cont.)
This gives \( x \lor y = (x \lor y) \land (z/u \lor w/u) \). Using lattice distributivity, \( x \lor y = x_1 \lor x_2 \lor y_1 \lor y_2 \), where

\[
\begin{align*}
x_1 &= x \land (z/u), \\
x_2 &= x \land (w/u), \\
y_1 &= y \land (z/u), \\
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Notice:

\[
\begin{align*}
x_1 \leq z/u & \implies u \leq x_1 \downarrow z \leq (x_1 \land y_2) \downarrow z, \\
x_2 \leq w/u & \implies u \leq x_2 \downarrow w \leq (x_2 \land y_1) \downarrow w, \\
y_1 \leq z/u & \implies u \leq y_1 \downarrow z \leq (x_2 \land y_1) \downarrow z, \\
y_2 \leq y/u & \implies u \leq y_2 \downarrow w \leq (x_1 \land y_2) \downarrow w.
\end{align*}
\]
This gives

\[ u \leq (x_1 \land y_2) \backslash (z \land w) \leq x_1 \lor z \lor y_2 \lor w \]

\[ u \leq (x_2 \land y_1) \backslash (z \land w) \leq x_2 \lor z \lor y_1 \lor w \]

\[ u \leq x_1 \lor z \leq x_1 \lor z \lor y_1 \lor w \]

\[ u \leq y_2 \lor w \leq x_2 \lor z \lor y_2 \lor w. \]
The general picture (cont.)

This gives

\[ u \leq (x_1 \land y_2) \land (z \land w) \leq x_1 \mid z \lor y_2 \mid w \]

\[ u \leq (x_2 \land y_1) \land (z \land w) \leq x_2 \mid z \lor y_1 \mid w \]

\[ u \leq x_1 \mid z \leq x_1 \mid z \lor y_1 \mid w \]

\[ u \leq y_2 \mid w \leq x_2 \mid z \lor y_2 \mid w. \]

Hence,

\[ u \leq (x_1 \mid z \lor y_2 \mid w) \land (x_2 \mid z \lor y_1 \mid w) \land (x_1 \mid z \lor y_1 \mid w) \land (x_2 \mid z \lor y_2 \mid w) \]

\[ = ((x_2 \mid z \land x_1 \mid z) \lor y_1 \mid w) \land ((x_1 \mid z \lor x_2 \mid z) \lor y_2 \mid w) \]

\[ = (x_1 \lor x_2) \mid z \lor (y_1 \mid w \land y_2 \mid w) \]

\[ = x \mid z \lor y \mid w. \]
By similar methods, we can obtain the following.

**Theorem (WF, P. Jipsen 2018):**

Let $A$ be a residuated binar whose lattice reduct is distributive.

- If $A$ satisfies both $(\lor/)\,\cap\,(\land\setminus)$, then $A$ also satisfies $(\setminus\lor)$.  
- If $A$ satisfies both $(\setminus\lor)\,\cap\,(\lor/)\,$, then $A$ also satisfies $(\lor/)$.  
- If $A$ satisfies both $(\land\cdot)\,\cap\,(\lor/)\,$, then $A$ also satisfies $(\lor/)$.  
- If $A$ satisfies both $(\land\cdot)\,\cap\,(\setminus\lor)\,$, then $A$ also satisfies $(\land\cdot)$.  
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- If $A$ satisfies both $(\land\setminus)\,\cap\,(\land\cdot)\,$, then $A$ also satisfies $(\land\cdot)$.
In general, there are no other implications among the nontrivial distributive laws.
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The general picture (cont.)

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- Both methods of proof depend on distributivity of the lattice reduct in crucial ways.
The role of distributivity

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What about the non-distributive case?
Some remarks:

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- We originally obtained the proofs of the implications between nontrivial distributive laws by duality-theoretic methods (and our interest in these equations stems from duality theory).
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What about the non-distributive case? Still open.
The countermodels above are all based on Boolean lattices.
Some more results

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**Proposition (WF, P. Jipsen 2018):**

Let $\mathbf{A}$ be a residuated binar with neutral element $e$. If $e$ has a complement $e'$ and $\mathbf{A}$ satisfies any one of the distributive laws $(\cdot \land), (\land \cdot), (\land \backslash), (\backslash \land)$, then $\mathbf{A}$ is integral (i.e., $\mathbf{A}$ satisfies $x \leq e$).
Some more results

Proof: We prove the claim for $(\cdot \land)$. 

Note that $ot = \top \cdot \bot = (e \lor e') (e \land e') = [(e \lor e')] e \land [(e \lor e')] e' = \top \land (e e' \lor (e' e')^2) = e' \lor (e' e')^2$.

Thus $e' = (e' e')^2 = \bot$, and hence that $\bot$ is a complement of $e$. It follows that $e = e \lor \bot = \top$. 

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Some more results

Proof: We prove the claim for $(\cdot \land)$. 

Note that

\[
\bot = T \cdot \bot \\
= (e \lor e')(e \land e') \\
= [(e \lor e')e] \land [(e \lor e')e'] \\
= T \land (ee' \lor (e')^2) \\
= e' \lor (e')^2
\]
Proof: We prove the claim for $(\cdot \land)$.

Note that

\[
\bot = T \cdot \bot = (e \lor e')(e \land e') = [(e \lor e')e] \land [(e \lor e')e'] = T \land (ee' \lor (e')^2) = e' \lor (e')^2
\]

Thus $e' = (e')^2 = \bot$, and hence that $\bot$ is a complement of $e$. It follows that $e = e \lor \bot = T$. 

Proposition (???):
Let $A$ be a complemented residuated binar with neutral element $e$. If $A$ is integral, then $\land$ and $\cdot$ coincide.
Some more results (cont.)

Proposition (???):
Let \( A \) be a complemented residuated binar with neutral element \( e \). If \( A \) is integral, then \( \land \) and \( \cdot \) coincide.

Corollary:
Let \( A \) be a complemented residuated binar with neutral element \( e \). If \( A \) satisfies any one of the distributive laws \((\cdot \land),(\land \cdot),(\land \backslash),(\backslash \land)\), then \( A \) is a Boolean algebra.
Concluding remarks

Work is still on-going, and many questions remain:
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- The non-distributive case is still open.
- The role of semilinearity in the absence of associativity and a multiplicative neutral element is also unknown.
- What is the join of two varieties axiomatized by a collection of nontrivial distributive laws?
Thank you!