A Constructive View of Weak Topologies on a Topos

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Outline

In this talk:

- We introduce the notion of (productive) weak topology on a topos $E$ and investigate some of its basic properties.
- We show that the set of all weak topologies on a complete topos $E$ is a complete residuated lattice.
- We give an explicit description of a restricted associated sheaf functor on a topos $E$ in two steps.
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In this talk:

- we introduce the notion of (productive) weak topology on a topos $\mathcal{E}$ and investigate some of its basic properties.

we show that the set of all weak topologies on a $(co)$complete topos $\mathcal{E}$ is a complete resituated lattice.

we give an explicit description of a restricted associated sheaf functor on a topos $\mathcal{E}$ in two steps.
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- we show that the set of all weak topologies on a (co)complete topos $\mathcal{E}$ is a complete resituated lattice.
In this talk:

- we introduce the notion of (productive) weak topology on a topos $\mathcal{E}$ and investigate some of its basic properties.
- we show that the set of all weak topologies on a (co)complete topos $\mathcal{E}$ is a complete resituated lattice.
- we give an explicit description of a restricted associated sheaf functor on a topos $\mathcal{E}$ in two steps.
(Elementary) topos

Definition

An (elementary) topos is a category $\mathcal{E}$ with finite limits, provided that the following conditions are satisfied:

1. $\mathcal{E}$ is cartesian closed, i.e. all objects of $\mathcal{E}$ are exponentiable;
2. $\mathcal{E}$ has a subobject classifier, that is, an object $\Omega$ equipped with a monomorphism $\text{true}: 1 \rightarrowtail \Omega$ such that, given any monomorphism $m: S \rightarrow B$ in $\mathcal{E}$; there is a unique map $\text{char}(m): B \rightarrow \Omega$ (sometimes denoted by $\text{char}(S)$) for which the following square is a pullback:
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$$
\begin{array}{ccc}
S & \rightarrow & 1 \\
\downarrow m & & \downarrow \text{true} \\
B & \rightarrow & \Omega.
\end{array}
$$
Internal Heyting algebra structure of $\Omega$

In fact, for each object $B$ of $\mathcal{E}$ we have a natural isomorphism in $B$ as follows

$$\text{Sub}_\mathcal{E}(B) \cong \text{Hom}_\mathcal{E}(B, \Omega)$$

The subobject classifier $\Omega$ on a topos $\mathcal{E}$ has an internal Heyting algebra structure. In details,
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The subobject classifier $\Omega$ on a topos $\mathcal{E}$ has an internal Heyting algebra structure. In details,

1. The meet operation $\cap : \text{Sub}_{\mathcal{E}}(B) \times \text{Sub}_{\mathcal{E}}(B) \to \text{Sub}_{\mathcal{E}}(B)$ is natural in $B$. Under the isomorphism $\text{Hom}_{\mathcal{E}}(B, \Omega) \cong \text{Sub}_{\mathcal{E}}(B)$, which is again natural in $B$, we obtain an operation $\wedge_B$ such that the following diagram is commutative:
Internal Heyting algebra structure of $\Omega$

$$\begin{align*}
\text{Sub}_E(B) \times \text{Sub}_E(B) & \xrightarrow{\cap} \text{Sub}_E(B) \\
\text{Hom}_E(B, \Omega) \times \text{Hom}_E(B, \Omega) & \xrightarrow{\land_B} \text{Hom}_E(B, \Omega)
\end{align*}$$

Since the operation $\land_B$ is natural in $B$, so by the Yoneda lemma $\land_B$ comes from a uniquely determined map $\land = \land_{\Omega} \times \Omega$ via composition which is $\land = \land_{\Omega} \times \Omega (\text{id}_\Omega \times \Omega)$. The arrow $\land$ is the internal meet operation on $\Omega$. 
Internal Heyting algebra structure of $\Omega$

Since the operation $\land_B$ is natural in $B$, so by the Yoneda lemma $\land_B$ comes from a uniquely determined map $\land : \Omega \times \Omega \to \Omega$ via composition which is $\land = \land_{\Omega \times \Omega}(\text{id}_{\Omega \times \Omega})$. The arrow $\land$ is the internal meet operation on $\Omega$. 
Similarly, we can define an internal join operation $\lor: \Omega \times \Omega \to \Omega$ and an internal implication operation $\Rightarrow: \Omega \times \Omega \to \Omega$ on $\Omega$. Under the isomorphism $\text{Sub}_E(1) \cong \text{Hom}_E(1, \Omega)$, the top and bottom elements of $\text{Sub}_E(1)$ which are $1 \hookrightarrow 1$ and $0 \hookrightarrow 1$, respectively, correspond to the internal top and internal bottom elements "true" $= \text{char}(1 \hookrightarrow 1)$ and "false" $= \text{char}(0 \hookrightarrow 1)$ of $\Omega$. 
Similarly, we can define an internal join operation \( \lor : \Omega \times \Omega \to \Omega \) and an internal implication operation \( \Rightarrow : \Omega \times \Omega \to \Omega \) on \( \Omega \).

Under the isomorphism \( \text{Sub}_E(1) \cong \text{Hom}_E(1, \Omega) \), the top and bottom elements of \( \text{Sub}_E(1) \) which are \( 1 \to 1 \) and \( 0 \to 1 \), respectively, correspond to the internal top and internal bottom elements "true = char(1 \to 1)" and "false = char(0 \to 1)" of \( \Omega \).
Definition

A weak Lawvere-Tierney topology (or a weak topology, for short) on a topos $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ such that:

1. $j \circ \text{true} = \text{true}$;
2. $j \circ \wedge \leq \wedge \circ (j \times j)$, in which $\leq$ stands for the order on $\Omega$.

Furthermore, $j$ is productive whenever the non-equality in (2) is an equality.

An idempotent weak topology on $\mathcal{E}$ is called a (Lawvere-Tierney) topology on $\mathcal{E}$. 
Weak topologies

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A Constructive View of Weak Topologies...
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Furthermore, \( j \) is productive whenever the non-equality in (2) is an equality.
弱拓扑

定义

一个弱 Lawvere-Tierney 拓扑（或简称弱拓扑）在拓扑学上 $E$ 是一个映射 $j : \Omega \to \Omega$ 且满足以下条件：

1. $j \circ \text{true} = \text{true}$；
2. $j \circ \land \leq \land \circ (j \times j)$，其中 $\leq$ 表示 $\Omega$ 上的顺序。

此外，$j$ 是 productivity 当且仅当条件 (2) 中的不等式成为等式。

一个 idempotent 的弱拓扑 $E$ 被称为 (Lawvere-Tierney) 拓扑。
Some Examples of Weak Topologies

Example

The composite of any two topologies on a topos $E$ is a productive weak topology. It is a topology on $E$ if and only if it is idempotent.

It is well known that the commutative monoid of natural endomorphisms of the identity functor on a topos $E$ is called the center of $E$. Let $\alpha$ be a natural endomorphism of the identity functor on $E$. It is easy to see that $\alpha \Omega$ is a productive weak topology on $E$. It will be a topology on $E$ if $\alpha^2 \Omega = \alpha \Omega$. 

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A Constructive View of Weak Topologies ...
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Modal closure operators

Definition
An operator on the subobjects of each object $E$ of $\mathcal{E}$

$$A \mapsto \overline{A}, \quad \text{Sub}_E(E) \rightarrow \text{Sub}_E(E),$$

is a modal closure operator if and only if it has, for all $A, B \in \text{Sub}_E(E)$, the properties:
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1. (Extension) $A \subseteq \overline{A}$;
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is a modal closure operator if and only if it has, for all $A, B \in \text{Sub}_{\mathcal{E}}(E)$, the properties:

1. (Extension) $A \subseteq \overline{A}$;
2. (Monotonicity) $A \subseteq B$ yields that $\overline{A} \subseteq \overline{B}$;
Modal closure operators

**Definition**
An operator on the subobjects of each object $E$ of $\mathcal{E}$

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is a modal closure operator if and only if it has, for all $A, B \in \text{Sub}_\mathcal{E}(E)$, the properties:

1. (Extension) $A \subseteq \overline{A}$;
2. (Monotonicity) $A \subseteq B$ yields that $\overline{A} \subseteq \overline{B}$;
3. (Modal) For each arrow $f : F \to E$ in $\mathcal{E}$, we have $\overline{f^{-1}(A)} = f^{-1}(\overline{A})$, where $f^{-1}$ is the pullback functor.
Weak Topologies and Modal Closure Operators

Any weak topology \( j \) on \( E \), determines a modal closure operator \( A \mapsto \overset{\rightarrow}{A} \) on the subobjects \( A \hookrightarrow E \) of each object \( E \), in such a way that for any subobject \( A \overset{\iota}{\hookrightarrow} E \), the \( j \)-closure of \( A \) is the subobject \( A \) of \( E \) with the characteristic map \( j \text{char}(\iota) \), shown as in the diagram below:

\[
\begin{array}{ccc}
A & \rightarrow & E \\
\downarrow & & \downarrow \\
\iota & \rightarrow & \Omega \\
\end{array}
\]
Any weak topology \( j \) on \( E \), determines a modal closure operator \( A \mapsto \overline{A} \) on the subobjects \( A \hookrightarrow E \) of each object \( E \), in such a way that for any subobject \( A \hookrightarrow E \), the \( j \)-closure of \( A \) is the subobject \( \overline{A} \) of \( E \) with the characteristic map \( j \text{char}(\iota) \), shown as in the diagram below.

\[
\begin{array}{ccc}
\overline{A} & \rightarrow & 1 \\
\downarrow & & \downarrow \\
\iota & \rightarrow & \text{true} \\
\downarrow & & \downarrow \\
E & \rightarrow & 1 \\
\downarrow & & \downarrow \\
\iota & \rightarrow & \text{true} \\
\downarrow & & \downarrow \\
\overline{A} & \rightarrow & \Omega \\
\downarrow & & \downarrow \\
\iota & \rightarrow & \text{char}(\iota) \\
\downarrow & & \downarrow \\
E & \rightarrow & \Omega \\
\end{array}
\]
Conversely, any modal closure operator on a topos $\mathcal{E}$ always gives a unique weak topology $j$ as indicated in the following pullback diagram:

$$
\begin{array}{c}
\text{true} \\
\downarrow \\
\Omega \\
\downarrow \\
\text{true}
\end{array}
\quad
\begin{array}{c}
\mathbb{1} \\
\downarrow \\
\Omega \\
\downarrow \\
\mathbb{1}
\end{array}
\quad
\begin{array}{c}
\downarrow \\
\Omega
\end{array}
$$

One can prove that

**Lemma**

On a topos $\mathcal{E}$, weak topologies $j$ are in one-to-one correspondence with modal closure operators $(\cdot)$.
Conversely, any modal closure operator on a topos $\mathcal{E}$ always gives a unique weak topology $j$ as indicated in the following pullback diagram:

\[
\begin{array}{ccc}
\bar{1} & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
\text{true} & \longrightarrow & \text{true} \\
\downarrow & & \downarrow \\
\Omega & \longrightarrow & \Omega \\
\end{array}
\]

One can prove that
Conversely, any modal closure operator on a topos $\mathcal{E}$ always gives a unique weak topology $j$ as indicated in the following pullback diagram:

\[
\begin{array}{ccc}
1 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
true & \swarrow & true \\
\Omega & \underset{j}{\longrightarrow} & \Omega \\
\end{array}
\]

One can prove that

**Lemma**

On a topos $\mathcal{E}$, weak topologies $j$ are in one-to-one correspondence with modal closure operators $\overline{\cdot}$. 
For a weak topology $j$ on $\mathcal{E}$, it can be easily checked that $j \circ \wedge = \text{char}(\text{true} \times \text{true})$ and $\wedge \circ (j \times j) = \text{char}(\text{true} \times \text{true})$. So two subobjects $1 \times 1$ and $1 \times 1$ of $\Omega \times \Omega$, are not equal. This means that the modal closure operator associated to $j$, is not productive; that is the closure does not commute with products. We can prove that for a weak topology $j$ on $\mathcal{E}$, the modal closure operator associated to $j$, is productive if and only if one has $j \circ \wedge = \wedge \circ (j \times j)$ if and only if the modal closure operator associated to $j$, commutes with binary intersections. For this reason, we call a weak topology $j$ with the property $j \circ \wedge = \wedge \circ (j \times j)$ a productive weak topology.
For a weak topology $j$ on $E$, it can be easily checked that $j \circ \land = \text{char}(\text{true} \times \text{true})$ and $\land \circ (j \times j) = \text{char}(\overline{\text{true}} \times \overline{\text{true}})$. So two subobjects $1 \times 1$ and $1 \times 1$ of $\Omega \times \Omega$, are not equal. This means that the modal closure operator associated to $j$, is not productive; that is the closure does not commute with products.
For a weak topology $j$ on $E$, it can be easily checked that 
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and 
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So two subobjects $1 \times 1$ and $\bar{1} \times \bar{1}$ of $\Omega \times \Omega$, are not equal. This means that the modal closure operator associated to $j$, is not \textit{productive}; that is the closure does not commute with products.
We can prove that for a weak topology $j$ on $E$, the modal closure operator associated to $j$, \textit{is productive} if and only if one has $j \circ \wedge = \wedge \circ (j \times j)$ if and only if the modal closure operator associated to $j$, commutes with binary intersections.
For a weak topology $j$ on $\mathcal{E}$, it can be easily checked that $j \circ \wedge = \text{char}(\text{true} \times \text{true})$ and $\wedge \circ (j \times j) = \text{char}(\text{true} \times \text{true})$. So two subobjects $1 \times 1$ and $\overline{1} \times \overline{1}$ of $\Omega \times \Omega$, are not equal. This means that the modal closure operator associated to $j$, is not productive; that is the closure does not commute with products.

We can prove that for a weak topology $j$ on $\mathcal{E}$, the modal closure operator associated to $j$, is productive if and only if one has $j \circ \wedge = \wedge \circ (j \times j)$ if and only if the modal closure operator associated to $j$, commutes with binary intersections.

For this reason, we call a weak topology $j$ with the property $j \circ \wedge = \wedge \circ (j \times j)$ a productive weak topology.
$j$-closed, $j$-dense, $j$-sheaf and $j$-separated

Definition

Let $j$ be a weak topology on $\mathcal{E}$ and $(\cdot\bar{\cdot})$ the modal closure operator associated to $j$. A monomorphism $k : A \hookrightarrow C$ in $\mathcal{E}$ is

$\text{unique}$

We say that $C$ is $j$-separated if the arrow $g$ exists and is unique.
$j$-closed, $j$-dense, $j$-sheaf and $j$-separated

**Definition**

Let $j$ be a weak topology on $\mathcal{E}$ and $(\cdot)$ the modal closure operator associated to $j$. A monomorphism $k : A \hookrightarrow C$ in $\mathcal{E}$ is

- **$j$-dense** whenever $\overline{A} = C$, as subobjects of $C$;

...
j-closed, j-dense, j-sheaf and j-separated

**Definition**

Let $j$ be a weak topology on $\mathcal{E}$ and $(\cdot)$ the modal closure operator associated to $j$. A monomorphism $k : A \rightarrowtail C$ in $\mathcal{E}$ is

- **$j$-dense** whenever $\overline{A} = C$, as subobjects of $C$;
- **$j$-closed** if $\overline{A} = A$, as subobjects of $C$.

Moreover, an object $C$ is called a **$j$-sheaf** whenever for any $j$-dense monomorphism $m : B \rightarrowtail A$, one can uniquely extend any arrow $h : B \rightarrow C$ in $\mathcal{E}$ as follows:

\[
\begin{array}{c}
B \\
\downarrow \\
A \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
C \\
\end{array}
\]

We say that $C$ is **$j$-separated** if the arrow $g$ exists, it is unique.
Definition

Let \( j \) be a weak topology on \( \mathcal{E} \) and \( (\cdot) \) the modal closure operator associated to \( j \). A monomorphism \( k : A \rightarrowtail C \) in \( \mathcal{E} \) is

- \( j \)-dense whenever \( \overline{A} = C \), as subobjects of \( C \);
- \( j \)-closed if \( \overline{A} = A \), as subobjects of \( C \).

Moreover, an object \( C \) is called a \( j \)-sheaf whenever for any \( j \)-dense monomorphism \( m : B \rightarrowtail A \), one can uniquely extend any arrow \( h : B \rightarrow C \) in \( \mathcal{E} \) to \( A \) as follows.
**j-closed, j-dense, j-sheaf and j-separated**

**Definition**

Let \( j \) be a weak topology on \( \mathcal{E} \) and \( (\cdot) \) the modal closure operator associated to \( j \). A monomorphism \( k : A \hookrightarrow C \) in \( \mathcal{E} \) is

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\[
\begin{array}{ccc}
B & \xrightarrow{h} & C \\
\downarrow m & \nearrow g & \\
A & &
\end{array}
\]
**j-closed, j-dense, j-sheaf and j-separated**

**Definition**

Let \( j \) be a weak topology on \( \mathcal{E} \) and \( (\bar{\cdot}) \) the modal closure operator associated to \( j \). A monomorphism \( k : A \rightarrowtail C \) in \( \mathcal{E} \) is

- **j-dense** whenever \( \bar{A} = C \), as subobjects of \( C \);
- **j-closed** if \( \bar{A} = A \), as subobjects of \( C \).

Moreover, an object \( C \) is called a **j-sheaf** whenever for any \( j \)-dense monomorphism \( m : B \rightarrowtail A \), one can uniquely extend any arrow \( h : B \rightarrow C \) in \( \mathcal{E} \) to \( A \) as follows

\[
\begin{array}{ccc}
B & \xrightarrow{h} & C \\
\downarrow{m} & & \downarrow{g} \\
A & & \\
\end{array}
\]

We say that \( C \) is **j-separated** if the arrow \( g \) exists, it is unique.
Subobject classifier of closed subobjects

For a weak topology $j$ on a topos $\mathcal{E}$,
Subobject classifier of closed subobjects

For a weak topology \( j \) on a topos \( \mathcal{E} \),

- We will denote the equalizer of two arrows \( j, \text{id}_\Omega : \Omega \to \Omega \) by \( \Omega_j \) as follows

\[
\Omega_j \xrightarrow{m} \Omega \xrightarrow{j} \Omega \xrightarrow{\text{id}_\Omega} \Omega.
\]
Subobject classifier of closed subobjects

For a weak topology $j$ on a topos $E$,

- We will denote the equalizer of two arrows $j, \text{id}_\Omega : \Omega \to \Omega$ by $\Omega_j$ as follows

$$\Omega_j \xrightarrow{m} \Omega \xrightarrow{j} \Omega \xrightarrow{\text{id}_\Omega} \Omega.$$ 

The object $\Omega_j$ classifies closed subobjects, in the sense that, for each object $E$ of $E$, there is a bijection

$$\text{Hom}_E(E, \Omega_j) \sim \text{ClSub}_E(E);$$

which is natural in $E$. Here $\text{ClSub}_E(E)$ is the set of all closed subobjects of $E$. 
For a weak topology $j$ on a topos $\mathcal{E}$,
Some Notations

For a weak topology $j$ on a topos $\mathcal{E}$,

- We will denote the full subcategories of $\mathcal{E}$ consisting of all $j$-separated objects and $j$-sheaves by $\text{Sh}_j(\mathcal{E})$ and $\text{Sep}_j(\mathcal{E})$, respectively. One can check that for a productive weak topology $j$ on $\mathcal{E}$, $\text{Sh}_j(\mathcal{E})$ is a topos with the subobject classifier $\Omega_j$.
Some Notations

For a weak topology $j$ on a topos $\mathcal{E}$,

- We will denote the full subcategories of $\mathcal{E}$ consisting of all $j$-separated objects and $j$-sheaves by $\text{Sh}_j(\mathcal{E})$ and $\text{Sep}_j(\mathcal{E})$, respectively. One can check that for a productive weak topology $j$ on $\mathcal{E}$, $\text{Sh}_j(\mathcal{E})$ is a topos with the subobject classifier $\Omega_j$.

- We will denote the image of the weak topology $j$ by $\text{im}(j)$, that is the smallest subobject $k : \text{im}(j) \hookrightarrow \Omega$ which $j$ can factor through it.
The following proposition is the basic difference between weak topologies and topologies on $\mathcal{E}$. It shows that we are unable to construct the associated sheaf functor to a weak topology $j$ on $\mathcal{E}$ as usual.
The following proposition is the basic difference between weak topologies and topologies on $E$. It shows that we are unable to construct the associated sheaf functor to a weak topology $j$ on $E$ as usual.

**Proposition**

For a weak topology $j$ on a topos $E$, we factor $j$ through its image as

$$\Omega \xrightarrow{r} \text{im}(j) \xrightarrow{k} \Omega.$$  

Then $j$ is idempotent (or equivalently, is a topology on $E$) if and only if $\Omega_j = \text{im}(j)$, as subobjects of $\Omega$.  

Remark

The set of (productive) weak topologies on a topos \( \mathcal{E} \) has a natural partial order given by \( j \leq k \) if and only if \( j = j \land k \), for any (productive) weak topologies \( j, k : \Omega \to \Omega \), where \( j \land k \) is the composite arrow \( \Omega \xrightarrow{(j,k)} \Omega \times \Omega \xrightarrow{\land} \Omega \).
Remark

- The set of (productive) weak topologies on a topos $\mathcal{E}$ has a natural partial order given by $j \leq k$ if and only if $j = j \wedge k$, for any (productive) weak topologies $j, k : \Omega \to \Omega$, where $j \wedge k$ is the composite arrow $\Omega \xrightarrow{(j,k)} \Omega \times \Omega \xrightarrow{\wedge} \Omega$.

- We denote by $\text{WTop}(\mathcal{E})$, $\text{PWTOP}(\mathcal{E})$ and $\text{Top}(\mathcal{E})$ for the posets of weak topologies, productive weak topologies and topologies on $\mathcal{E}$, respectively. It is clear that

$$\text{Top}(\mathcal{E}) \subseteq \text{PWTOP}(\mathcal{E}) \subseteq \text{WTop}(\mathcal{E}).$$

Notice that all these posets have the same binary meets which is pointwise, and also they have the top and bottom elements which are $\text{true} \circ \text{!}_\Omega$ and $\text{id}_\Omega$, respectively.
The Residuated Lattice of Weak Topologies

It is clear that \((\text{WT}op(\mathcal{E}), \circ, \text{id}_\Omega)\) is a monoid in which \(\circ\) is the ordinary composition of weak topologies on \(\mathcal{E}\).
The Residuated Lattice of Weak Topologies

It is clear that \((\text{WTop}(\mathcal{E}), \circ, \text{id}_\Omega)\) is a monoid in which \(\circ\) is the ordinary composition of weak topologies on \(\mathcal{E}\).

Let \(\mathcal{E}\) be a (co)complete topos. We define two binary operations \(\setminus\) and \(\div\) on \(\text{WTop}(\mathcal{E})\) given by

\[
   j \setminus k = \bigwedge \{j' | j' \in \text{WTop}(\mathcal{E}), j \circ j' \geq k\},
\]

and

\[
   k \div j = \bigwedge \{j' | j' \in \text{WTop}(\mathcal{E}), j' \circ j \geq k\},
\]

for weak topologies \(j\) and \(k\) on \(\mathcal{E}\).
The Residuated Lattice of Weak Topologies

It is clear that \(( \text{WTop}(\mathcal{E}), \circ, \text{id}_{\Omega} )\) is a monoid in which \(\circ\) is the ordinary composition of weak topologies on \(\mathcal{E}\).

Let \(\mathcal{E}\) be a (co)complete topos. We define two binary operations \(\setminus\) and \(/\) on \(\text{WTop}(\mathcal{E})\) given by

\[
 j \setminus k = \bigwedge \{ j' | j' \in \text{WTop}(\mathcal{E}), \ j \circ j' \geq k \},
\]

and

\[
 k / j = \bigwedge \{ j' | j' \in \text{WTop}(\mathcal{E}), \ j' \circ j \geq k \},
\]

for weak topologies \(j\) and \(k\) on \(\mathcal{E}\). It is easily seen that we have

\[
 j \circ j' \geq k \iff j \geq k / j' \iff j' \geq j \setminus k.
\]

The desired result now is:
The Residuated Lattice of Weak Topologies

It is clear that \((\text{WTop}(\mathcal{E}), \circ, \text{id}_\Omega)\) is a monoid in which \(\circ\) is the ordinary composition of weak topologies on \(\mathcal{E}\).

Let \(\mathcal{E}\) be a (co)complete topos. We define two binary operations \(\setminus\) and \(/\) on \(\text{WTop}(\mathcal{E})\) given by

\[
 j \setminus k = \bigwedge \{ j'| j' \in \text{WTop}(\mathcal{E}), \ j \circ j' \geq k \},
\]

and

\[
 k / j = \bigwedge \{ j'| j' \in \text{WTop}(\mathcal{E}), \ j' \circ j \geq k \},
\]

for weak topologies \(j\) and \(k\) on \(\mathcal{E}\). It is easily seen that we have

\[
 j \circ j' \geq k \iff j \geq k / j' \iff j' \geq j \setminus k.
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The desired result now is:

**Theorem**

Let \(\mathcal{E}\) be a (co)complete topos. Then, \((\text{WTop}(\mathcal{E}), \land, \lor, \circ, \text{id}_\Omega, \setminus, /)\) is a complete residuated lattice.
Idempotent Hull of a (Productive) Weak Topology

Let $\mathcal{E}$ be a cocomplete topos and $j$ a (productive) weak topology on $\mathcal{E}$. We define the ascending extended ordinal chain of (productive) weak topologies

$$j \leq j^2 \leq j^3 \leq \ldots \leq j^\alpha \leq j^{\alpha+1} \leq \ldots \leq j^\infty \leq j^{\infty+1}$$

in which:

$$j^{\alpha+1} = j \circ j^\alpha, \quad j^\beta = \bigvee_{\gamma<\beta} j^\gamma$$

for every (small) ordinal number $\alpha$ and for $\alpha = \infty$, and for every limit ordinal $\beta$ and for $\beta = \infty$; here $\infty, \infty + 1$ are (new) elements with $\infty + 1 > \infty > \alpha$ for all $\alpha \in \text{Ord}$, the class of small ordinals. Then,
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**Proposition**

Let $j$ be a (productive) weak topology on a cocomplete topos $\mathcal{E}$. Then $j^\infty$ is the idempotent hull of $j$, i.e. the smallest topology containing $j$. Moreover, one has $\text{Sh}_{j^\infty}(\mathcal{E}) = \bigcap_{\gamma < \infty} \text{Sh}_{j^\gamma}(\mathcal{E})$, as full subcategories of $\mathcal{E}$. 
Weak Topologies and Join of a Set of Topologies

For a cocomplete topos $\mathcal{E}$, the inclusion functor

$$U : \text{Top}(\mathcal{E}) \hookrightarrow \text{WTop}(\mathcal{E})$$

(or $U : \text{Top}(\mathcal{E}) \hookrightarrow \text{PWTop}(\mathcal{E})$) has a left adjoint,

$$F : \text{WTop}(\mathcal{E}) \rightarrow \text{Top}(\mathcal{E})$$

which, as any left adjoint to an inclusion, assigns to each (productive) weak topology $j$ the least topology $j^\infty$ with $j \leq j^\infty$, we call it the topological reflection of $j$.

Thus, the join of a set of topologies $\{j_\alpha\}_{\alpha \in \Lambda}$ on $\mathcal{E}$ is the topological reflection of its join in $\text{WTop}(\mathcal{E})$, i.e. $(\bigvee_{\alpha \in \Lambda} U(j_\alpha))^\infty$.

Note that for a cocomplete topos $\mathcal{E}$, the subobject classifier $\Omega$ has arbitrary joins.
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Some Notations

Let $j$ be a weak topology on $E$ and $E$ an object of $E$. We assume that $\Omega \hookrightarrow \text{im}(j)$ and $\Theta \hookrightarrow \Omega$ be the image factorization of $j$ and $E \twoheadrightarrow S \hookrightarrow \text{im}(j)$ be the image factorization of the compound arrow $E \{\cdot\} E \rightarrow \text{im}(j)$ in which $\{\cdot\}_E : E \hookrightarrow \Omega$ stands for the transpose of the characteristic map of the diagonal $\Delta_E : E \rightarrow E \times E$ which is the arrow $(\text{id}_E, \text{id}_E)$.

For a productive weak topology $j$ on a topos $E$, we write $C_j$ for the full subcategory of $E$ consisting of all objects $E$ of $E$ for which the subobject $\Delta_E$ of $E \times E$ is closed.
Let $j$ be a weak topology on $\mathcal{E}$ and $E$ an object of $\mathcal{E}$. We assume that $\Omega \xrightarrow{r} \text{im}(j) \xleftarrow{k} \Omega$ be the image factorization of $j$ and $E \xrightarrow{\theta_E} S_E \xrightarrow{\omega_E} \text{im}(j)^E$ the image factorization of the compound arrow $r^E \{\cdot\}_E : E \rightarrow \text{im}(j)^E$ in which $\{\cdot\}_E : E \xrightarrow{} \Omega^E$ stands for the transpose of the characteristic map of the diagonal $\triangle_E : E \rightarrow E \times E$ which is the arrow $(\text{id}_E, \text{id}_E)$. 
Some Notations

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- For a productive weak topology $j$ on a topos $\mathcal{E}$, we write $C_j$ for the full subcategory of $\mathcal{E}$ consisting of all objects $E$ of $\mathcal{E}$ for which the subobject $\overline{\triangle}_E$ of $E \times E$ is closed.
Characterization of the objects of $C_j$

The following characterizes the objects of $C_j$ for a weak topology $j$ on $E$. 
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**Lemma**

For a weak topology $j$ on a topos $E$ and for any object $E$ of $E$ the following are equivalent:

(i) $E$ is separated;

(ii) the diagonal $\Delta E \in \text{Sub}_E(E \times E)$ is a closed subobject of $E \times E$;

(iii) $j_\cdot \circ \{\cdot\}_E = \{\cdot\}_E$, as in the commutative diagram

$E \xrightarrow{\{\cdot\}_E} \Omega E \xleftarrow{j_\cdot} \Omega E$;

(iv) for any $f: A \to E$, the graph of $f$ which is $(\text{id}_A, f): A \to A \times E$, is a closed subobject of $A \times E$. 

Zeinab Khanjanzadeh, Ali Madanshekaf (Semnan University)
A Constructive View of Weak Topologies...
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\[
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E & \xrightarrow{\{\cdot\}_E} & \Omega_E \\
\downarrow & & \downarrow \\
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\end{array}
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![Diagram]

Zeinab Khanjanzadeh, Ali Madanshekaf

A Constructive View of Weak Topologies ...

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Now, we will give an explicit description of a restricted associated sheaf functor on $\mathcal{E}$ in two steps:

First, one can deduce that:

**Theorem** For any productive weak topology $j$ on a topos $\mathcal{E}$, the inclusion functor $\text{Sep}^j(\mathcal{E}) \hookrightarrow \mathcal{C}^j$ has a left adjoint $L: \mathcal{C}^j \to \text{Sep}^j(\mathcal{E})$ defined by $E \mapsto \mathcal{S}^E$.

Let $E$ be a complete, cocomplete and well-copowered topos and $j$ a productive weak topology on $\mathcal{E}$. Then, it is well known that the inclusion functor $\text{Sep}^j(\mathcal{E}) \hookrightarrow \mathcal{E}$ has a left adjoint $R: \mathcal{E} \to \text{Sep}^j(\mathcal{E})$.

One can construct the functor $R$ via the adjoint functor theorem.
First Part of the Restricted Associated Sheaf Functor

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- Let $\mathcal{E}$ be a complete, cocomplete and well-copowered topos and $j$ a productive weak topology on $\mathcal{E}$. Then, it is well known that the inclusion functor $\text{Sep}_j(\mathcal{E}) \hookrightarrow \mathcal{E}$ has a left adjoint $R : \mathcal{E} \longrightarrow \text{Sep}_j(\mathcal{E})$. One can construct the functor $R$ via the adjoint functor theorem.
Let $j$ be a weak topology on a topos $\mathcal{E}$. For any separated object $E$ of $\mathcal{E}$ the diagonal $\triangle_E$ is a closed subobject of $E \times E$. In this case, the characteristic map of $\triangle_E$ denoted by $\delta_E : E \times E \to \Omega$ satisfies $j\delta_E = \delta_E$. Since $\Omega_j$ is the equalizer of $j$ and $\text{id}_\Omega$, so there is a unique arrow

$$\alpha_E : E \times E \to \Omega_j \quad \text{s.t.} \quad m\alpha_E = \delta_E.$$ 

We denote the exponential transpose of $\alpha_E$ by $\hat{\alpha}_E : E \to \Omega^E_j$. Then we have:
Some Notation

Let $j$ be a weak topology on a topos $\mathcal{E}$. For any separated object $E$ of $\mathcal{E}$ the diagonal $\triangle_E$ is a closed subobject of $E \times E$. In this case, the characteristic map of $\triangle_E$ denoted by $\delta_E : E \times E \to \Omega$ satisfies $j\delta_E = \delta_E$. Since $\Omega_j$ is the equalizer of $j$ and $\text{id}_\Omega$, so there is a unique arrow

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We denote the exponential transpose of $\alpha_E$ by $\hat{\alpha}_E : E \to \Omega^E_j$. Then we have:

**Lemma**

Let $j$ be a weak topology on a topos $\mathcal{E}$ and $E$ a separated object of $\mathcal{E}$. Then the arrow $\hat{\alpha}_E$ as defined before, is a monomorphism.
Second Part of the Restricted Associated Sheaf Functor

In what follows we provide the sheaf associated to a separated object in a topos $\mathcal{E}$.
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In what follows we provide the sheaf associated to a separated object in a topos \( \mathcal{E} \).

**Theorem**

Let \( j \) be a productive weak topology on a topos \( \mathcal{E} \) and \( E \) a separated object of \( \mathcal{E} \). Let \( \overline{E} \) be the closure of \( E \) as a subobject of \( \Omega^E_j \) via the arrow \( \hat{\alpha}_E : E \hookrightarrow \Omega^E_j \). Then \( \overline{E} \) is a \( j \)-sheaf in \( \mathcal{E} \).

Moreover, the inclusion functor \( \text{Sh}_j(\mathcal{E}) \hookrightarrow \text{Sep}_j(\mathcal{E}) \) has a left adjoint \( S : \text{Sep}_j(\mathcal{E}) \rightarrow \text{Sh}_j(\mathcal{E}) \) defined by \( E \mapsto \overline{E} \) as a subobject of \( \Omega^E_j \).
In what follows we provide the associated sheaf functor with respect to the weak topology $j$ on $\mathcal{E}$. 

Corollary: We can constitute the compound left adjoint $\mathbf{SL}: \mathbf{C}_j \rightarrow \mathbf{Sh}_j(\mathcal{E})$ to the inclusion functor $\mathbf{Sh}_j(\mathcal{E}) \rightarrowtail \mathbf{C}_j$ which assigns to any $E$ of $\mathbf{C}_j$ the sheaf $S_E$ as a subobject of $\Omega^{S_E}_j$. 

In the case of a complete, cocomplete and well-copowered topos $\mathcal{E}$ the inclusion functor $\mathbf{Sh}_j(\mathcal{E}) \rightarrowtail \mathcal{E}$ has the compound left adjoint $\mathbf{SR}: \mathcal{E} \rightarrow \mathbf{Sh}_j(\mathcal{E})$. 

Zeinab Khanjanzadeh, Ali Madanshekaf (Semnan University)
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**Corollary**

We can constitute the compound left adjoint $SL : \mathcal{C}_j \rightarrow \mathbf{Sh}_j(\mathcal{E})$ to the inclusion functor $\mathbf{Sh}_j(\mathcal{E}) \hookrightarrow \mathcal{C}_j$ which assigns to any $E$ of $\mathcal{C}_j$ the sheaf $\overline{S}_E$ as a subobject of $\Omega_{\mathcal{C}_j}^{S_E}$. 
Conclusion

In what follows we provide the associated sheaf functor with respect to the weak topology \( j \) on \( \mathcal{E} \).

**Corollary**

We can constitute the compound left adjoint \( SL : C_j \rightarrow \mathbf{Sh}_j(\mathcal{E}) \) to the inclusion functor \( \mathbf{Sh}_j(\mathcal{E}) \hookrightarrow C_j \) which assigns to any \( E \) of \( C_j \) the sheaf \( \overline{S}_E \) as a subobject of \( \Omega_j^{\overline{S}_E} \).

In the case of a complete, cocomplete and well-copowered topos \( \mathcal{E} \) the inclusion functor \( \mathbf{Sh}_j(\mathcal{E}) \hookrightarrow \mathcal{E} \) has the compound left adjoint \( SR : \mathcal{E} \rightarrow \mathbf{Sh}_j(\mathcal{E}) \).
Sheaves in the category of separated objects

We can define a weak topology (modal closure operator) on a category with finite limits. Hence, for a weak topology \( j \) on a topos \( \mathcal{E} \), the notion of \( j \)-sheaves can be defined in the finite complete category \( \text{Sep}_j(\mathcal{E}) \).
We can define a **weak topology (modal closure operator)** on a category with finite limits. Hence, for a weak topology \( j \) on a topos \( E \), the notion of \( j \)-sheaves can be defined in the finite complete category \( \text{Sep}_j(E) \). The following determines \( j \)-sheaves in \( \text{Sep}_j(E) \).
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**Proposition**

Let $j$ be a productive weak topology on $\mathcal{E}$ and $E$ a separated object of $\mathcal{E}$. Then the following conditions are equivalent:

1. $E$ is a $j$-sheaf in $\text{Sep}_j(\mathcal{E})$;
2. $E$ is a $j$-sheaf in $\mathcal{E}$;
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Thank You For Your Attention!