The logic of $\Sigma$ formulas

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The meaning of a proposition is the method of its verification.
- Moritz Schlick, in *Meaning and Verification*

Say: a proposition is **verifiable** iff it admits a method of verification. The negation of a verifiable proposition need not be verifiable.

**examples (ignoring experimental error)**

1. “$a^3 + b^3 = c^3$ is solvable” vs “$a^3 + b^3 = c^3$ is not solvable”
2. “ZFC is inconsistent” vs “ZFC is consistent”
3. “$\alpha \neq 1/137$” vs “$\alpha = 1/137$”

Verifiable propositions are closed under conjunction and disjunction. Verifiable propositions are not closed under negation.
propositional positivistic logic

**connectives**

\[ \land \quad \lor \quad \top \quad \bot \]

**definition**

A positivistic propositional theory \( T \) consists of implications \( \alpha \implies \beta \), with \( \alpha \) and \( \beta \) positivistic. A valuation \( m \) models \( T \) just in case \( m \models \alpha \) implies \( m \models \beta \), for all \( \alpha \implies \beta \) in \( T \).

**completeness theorem**

Let \( T \) be a positivistic propositional theory, and let \( \phi \implies \psi \) be an implication between positivistic formulas. Then, \( T \) proves \( \phi \Rightarrow \psi \) if and only if every valuation that models \( T \) also models \( \phi \Rightarrow \psi \).

Proof systems: Hilbert style, sequent calculus, etc.
For $\phi$ and $\psi$ positivistic formulas in verifiable propositional constants:

- $\phi$ and $\psi$ are always both verifiable
- $\phi \Rightarrow \psi$ is generally not verifiable

A deduction establishes nothing unless it consists of meaningful formulas.

To a finitist: “meaningful” means “verifiable by finite computation”
To a realist: “meaningful” means “verifiable by transfinite computation”
(Tarski’s definition of truth.)

An axiom $\alpha \Rightarrow \beta$ of should be interpreted as a rule of inference.
Example.

propositional constants $P, Q_1, Q_2, R$.

theory $T = \{P \Rightarrow Q_1 \lor Q_2, Q_1 \Rightarrow R, Q_2 \Rightarrow \bot\}$.

A deduction from $P$ to $R$:

- $P$
- $Q_1 \lor Q_2$
- $R \lor Q_2$
- $R \lor \bot$
- $R \lor R$
- $R$

We define the deductive system $\text{RK}(T)$. For each implication $\alpha \Rightarrow \beta$ that is a logical axiom or an axiom of $T$, we have the rule:

\[
\begin{align*}
\Phi(\alpha) \\
\hline
\Phi(\beta)
\end{align*}
\]
logical axioms of propositional positivistic logic

\[ \bot \Rightarrow \psi \quad \phi \Rightarrow T \]
\[ \phi \land \psi \Rightarrow \phi \quad \phi \land \psi \Rightarrow \psi \quad \phi \Rightarrow \phi \land \phi \]
\[ \psi \lor \psi \Rightarrow \psi \quad \phi \Rightarrow \phi \lor \psi \quad \psi \Rightarrow \phi \lor \psi \]
\[ (\phi \lor \psi) \land \chi \Rightarrow (\phi \land \chi) \lor (\psi \land \chi) \]

**Theorem (K)**

For positivistic propositional theories \( T \), and implications \( \phi \Rightarrow \psi \), TFAE:

1. every valuation modeling \( T \) models \( \phi \Rightarrow \psi \)
2. there is a classical derivation of \( T \vdash \phi \Rightarrow \psi \)
3. there is an intuitionistic derivation of \( T \vdash \phi \Rightarrow \psi \)
4. there is a deduction of \( \phi \Rightarrow \psi \) in \( RK(T) \)
5. every bounded distributive lattice modeling \( T \) models \( \phi \leq \psi \)
A coherent theory $T$ is consist of implications between coherent formulas.

$$\phi(t, \overline{w}) \implies \exists v: \phi(v, \overline{w})$$

$$\exists v: \psi(\overline{w}) \implies \psi(\overline{w})$$

$$\phi(\overline{w}) \land \exists v: \psi(v, \overline{w}) \implies \exists v: \phi(\overline{w}) \land \psi(v, \overline{w})$$

We define the deductive system $RK_{\exists}(T)$. For each implication $\alpha \Rightarrow \beta$ that is a logical axiom or an axiom of $T$, we have the rule:

$$\Phi(\alpha(t_1, \ldots, t_n))$$

$$\Phi(\beta(t_1, \ldots, t_n))$$
Theorem (K)

For coherent theories $T$, and implications $\phi \Rightarrow \psi$, TFAE:

1. every model of $\overline{T}$ models $\phi \Rightarrow \psi$
2. there is a classical derivation of $\overline{T} \vdash \phi \Rightarrow \psi$
3. there is an intuitionistic derivation of $\overline{T} \vdash \phi \Rightarrow \psi$
4. there is a deduction of $\phi \Rightarrow \psi$ in $\text{RK}_\exists(T)$

The free variables in a deduction can be treated as constant symbols. Compare to the induction rule of primitive recursive arithmetic:

$$
\frac{\phi(0) \quad \phi(v, w) \Rightarrow \phi(S(v), w)}{\phi(v', w)}
$$
Universal quantification is suspect whenever we have an infinite universe.

“potential infinity” vs. “completed infinity”

Even if the natural numbers form a completed totality, the totality of all sets need not be. (Russell’s paradox.)

In the computational framework: A class is “surveyable” just in case there is a (transfinite) process that surveys, i. e., sorts through the totality. (Weaver)

Similarly, a class is “definite” just in case quantification takes bivalent propositions to bivalent proposition. (Feferman)

**completed** ≈ **surveyable** ≈ **definite** ≈ realist universal quantification
universal quantification

\[ \psi(w) \implies \forall v: \psi(w) \quad \forall v: \phi(v, w) \implies \phi(t, w) \]

We define the system \( RI_{\exists, \forall} \) analogously to \( RK_{\exists} \).

**Theorem (K)**

Let \( T \) be a theory in coherent logic with universal quantification. For each implication \( \phi \Rightarrow \psi \), TFAE:

1. There is an intuitionistic derivation of \( T \vdash \phi \Rightarrow \psi \)
2. There is a deduction of \( \phi \Rightarrow \psi \) in \( RI_{\exists, \forall}(T) \)

We do not have a completeness result for \( RI_{\exists, \forall} \)!
surveyability axiom

We express realist universal quantification:

\[ \forall v : \phi(\bar{w}) \lor \psi(v, \bar{w}) \implies \phi(\bar{w}) \lor \forall v : \psi(v, \bar{w}) \]

If we can survey the universe, then we can either verify \( \phi(\bar{w}) \) or verify \( \psi(v, \bar{w}) \) for each value of \( v \).

We define the deductive system \( \text{RK}_{\exists, \forall}(T) \) to be the system \( \text{RI}_{\exists, \forall}(T) \) together with the above schema of logical axioms.

**Theorem (K)**

Let \( T \) be a theory in coherent logic with universal quantification. For each implication \( \phi \implies \psi \), TFAE:

1. every model of \( \bar{T} \) models \( \phi \implies \psi \)
2. there is a classical derivation of \( \bar{T} \vdash \phi \implies \psi \)
3. there is a deduction of \( \phi \implies \psi \) in \( \text{RK}_{\exists, \forall}(T) \)
In applications, our primitive predicates are usually decidable.

\[ \top \Rightarrow P(w) \lor \tilde{P}(w) \quad P(w) \land \tilde{P}(w) \Rightarrow \bot \]

In this case, \( RK_{\exists, \forall}(T) \) is essentially a classical system.

classical logic \( \iff \) decidable primitive predicates \( \land \) surveyable universe

In applications, the universe is usually not surveyable/definite/completed, but the universe is rather the union of surveyable subclasses. (sets)

\[ \forall v: \leadsto \forall v \in t: \]
The class of $\Sigma$ formulas is the closure of the atomic formulas for the forms

$$\phi \land \psi \quad \phi \lor \psi \quad \exists v: \phi \quad \forall v \in t: \psi$$

The system $\text{RK}_\Sigma(T)$ has the axiom schemes as $\text{RK}_\exists(T)$ together with the following logical axioms:

$$\top \Rightarrow x \in y \lor x \not\in y \quad x \in y \land y \not\in x \Rightarrow \bot$$

$$\psi(w) \Rightarrow \forall v \in t: \psi(w) \quad \forall v \in t: \phi(v, w) \Rightarrow \phi(s, w)$$

$$\forall v \in t: \phi(w) \lor \psi(v, w) \Rightarrow \phi(w) \lor \forall v \in t: \psi(v, w)$$

$$\forall v \in t: v \not\in t \lor \phi(v, w) \Rightarrow \forall v \in t: \phi(v, w)$$
Theorem (K)

Let $T$ be a set of implications between $\Sigma$ formulas, and let $\phi$ and $\psi$ be $\Sigma$ formulas. TFAE:

1. every model of $\overline{T}$ is a model of $\overline{\phi \Rightarrow \psi}$
2. there is a classical deduction of $\overline{T} \vdash \overline{\phi \Rightarrow \psi}$.
3. there is a deduction of $\phi \Rightarrow \psi$ in $\text{RK}_\Sigma(T)$.

The equivalence $(2) \iff (3)$ is a theorem of $I\Sigma_1$. 
In applications, equality is usually available.

1. $\top \implies x = x$
2. $x = y \implies y = x$
3. $x = y \land y = z \implies x = z$
4. $\phi(x, \overline{w}) \land x = y \implies \phi(y, \overline{w})$
5. $\top \implies x = y \lor x \neq y$
6. $x = y \lor y \neq x \implies \bot$
\((\forall z \in x: z \in y) \land (\forall z \in y: z \in x) \implies x = y\)

2. \(z \in \{x, y\} \iff z = x \lor z = y\)

3. \(z \in \bigcup x \iff \exists y \in x: z \in x\)

4. \(y \in \wp(x) \iff \forall z \in y: z \in x\)

5. \(T \implies \exists Y \in \wp(X): \forall x \in X: x \in Y \leftrightarrow \phi\)

6. \(\exists z \in x: T \implies \exists z \in x: \forall y \in x: z \notin y\)

7. \(\forall x \in X: \exists y: \phi \implies \exists Y: \forall x \in X: \exists y \in Y: \phi\)

8. \(T \implies \exists x: \text{Ord}(x) \land x \approx z\)

9. \(T \implies \exists V: \text{Mod}(V) \land z \in V\)
subuniverses

Mod(\(V\)) abbreviates:

\((\forall x \in V : \forall y \in x : y \in V) \land (\forall x \in V : \forall y \in V : \{x, y\} \in V) \land (\forall x \in V : \bigcup x \in V) \land (\forall x \in V : \wp(x) \in V) \land (\forall x \in V : \forall y \in \wp(V) : x \approx y \rightarrow y \in V)\)

corollary*
The deductive system \(RK_\Sigma(T_1)\) deduces

\(\text{Mod}(V) \Rightarrow \phi^V\)

for every axiom \(\phi\) of ZFC.
Well-known \( \Sigma \) truth predicate for \( \Sigma \) formulas:

\[ T(\phi \land \psi) \iff \exists w : \text{Wit}(w, \phi) \]

1. \( T(\phi \land \psi) \iff T(\phi) \land T(\psi) \)
2. \( T(\phi \lor \psi) \iff T(\phi) \lor T(\psi) \)
3. \( T(\exists v : \psi(v)) \iff \exists a : T(\psi(a)) \)
4. \( T(\forall v \in b : \phi(v)) \iff \forall a \in b : T(\phi(a)) \)
5. \( T(P(t_1, \ldots, t_n)) \)
   \[ \iff \exists a_1 : \cdots \exists a_n : P(a_1, \ldots, a_n) \land T(a_1 = t_1) \land \cdots \land T(a_n = t_n) \]
6. \( T(a = F(t_1, \ldots, t_n)) \)
   \[ \iff \exists b_1 : \cdots \exists b_n : a = F(b_1, \ldots, b_n) \land T(b_1 = t_1) \land \cdots \land T(b_n = t_n) \]
Define the theory $T_2$ to be the theory $T_1$ together with the axiom

$$T(\phi) \land D_{T_1}(\phi \Rightarrow \psi) \implies T(\psi)$$

**proposition (K)**

The theory $T_2$ is finitely axiomatizable, and $RK_{\Sigma}(T_2)$ deduces

$$T(\phi) \land D_{T_2}(\phi \Rightarrow \psi) \implies T(\psi).$$

Thus, $T_2$ describes a self-contained incompletable universe of pure sets, that includes nested transitive models of ZFC. The object language and the metalanguage coincide.
The primitive recursive set functions (Jensen and Karp) are obtained using the following recursion scheme:

\[ F(v, w) = H \left( \bigcup \{ F(r, w) \mid r \in v \}, v, w \right) \]

Rathjen defined a theory PRS of primitive recursive set functions, analogous to Skolem’s theory PRA of primitive recursive functions. Both PRS and PRA can be formulated as positivistic theories.

Define a positivistic theory \( \text{PRS}^+ \) by adding to PRS the \( \Sigma \)-collection schema, and the well-ordering principle.

\( \text{PRS}^+ \) is essentially a weak fragment of KP + WOP. It is strong enough to formalize elementary constructions.
Introduction Rules for $\land$ and $\lor$

\[
\frac{(\exists \phi \in K) \quad \Gamma, \phi \vdash \Delta}{\Gamma, \land K \vdash \Delta}
\]  \hspace{2cm}
\frac{(\forall \phi \in K) \quad \Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, \land K}
\]

\[
\frac{(\forall \phi \in K) \quad \Gamma, \phi \vdash \Delta}{\Gamma, \lor K \vdash \Delta}
\]  \hspace{2cm}
\frac{(\exists \phi \in K) \quad \Gamma \vdash \Delta, \phi}{\Gamma \vdash \Delta, \lor K}
\]

Cut Rule

\[
\frac{\Gamma \vdash \Delta, \phi \quad \Gamma', \phi \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
\]
model completeness principle

Let $T$ be a set of $\mathcal{L}_K^\infty\omega$ formulas. Conclude that $T$ is inconsistent for $\mathcal{L}_K^\infty\omega$ or that $T$ has a model.

set completeness principle

Let $A$ be a transitive set. Let $T$ be a set of $\mathcal{L}_K^\infty\omega$ formulas in the vocabulary $\{=,\in,S\}$ and parameters from $A$. Assume that $T$ includes basic axioms describing the structure $(A,=,\in)$. Conclude that $T$ is inconsistent for $\mathcal{L}_K^\infty\omega$ or that there exists a set $B \subseteq A$ such that $(A,=,\in,B)$ is a model of $T$.

theorem (K)

Work in PRS$^+$ together with cut-elimination* for $\mathcal{L}_K^\infty\omega$. Each may be proved using the other as an axiom:

1. the model completeness principle
2. the set completeness principle together with the axiom of infinity