Homeomorphisms of Čech-Stone remainders: the zero-dimensional case

Paul McKenney
Joint work with Ilijas Farah

BLAST 2018
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*If \( X^\ast \simeq Y^\ast \), how similar do \( X \) and \( Y \) have to be?*

**Theorem (Parovicenko)**

*Assume the Continuum Hypothesis. Then for every zero-dimensional, locally compact, noncompact, Hausdorff space \( X \), \( X^\ast \simeq \omega^\ast \).*
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Sketch.

Let $C(X)$ denote the Boolean algebra of clopen subsets of $X$, and $K(X)$ its ideal of compact-open sets. Then by Stone duality, it's enough to prove that $C(X)/K(X) \cong P(\omega)/\text{fin.}$ These Boolean algebras are both countably saturated and have size $c = \aleph_1$. A back-and-forth argument finishes the proof.
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**Theorem (Farah-McKenney)**

Assume OCA and $\text{MA}_{\aleph_1}$. Let $X$ and $Y$ be zero-dimensional, locally compact Polish spaces, and suppose $\varphi : X^* \to Y^*$ is a homeomorphism. Then there are cocompact subsets of $X$ and $Y$ which are homeomorphic, and moreover $\varphi$ is induced by such a homeomorphism.
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This says that under OCA and MA$_{\aleph_1}$, Čech-Stone remainders (in a certain class) are very *rigid*. 
Notation: given a set $V$ we write $[V]^2$ for the set of unordered pairs $\{v, w\} \ (v \neq w)$ of elements of $V$. 
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- there is a partition $V = \bigcup_{n=1}^{\infty} B_n$ such that for all $n$, $[B_n]^2 \cap G = \emptyset$. (*$G$ is countably chromatic.*)
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Note: the set-theoretic strength of OCA is in the “for every separable metric $V$” part. For instance, OCA is true in ZFC for analytic $V \subseteq \mathbb{R}$ (by a Cantor-Bendixon style argument).
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**Theorem (Veličković, 1993)**

*Assume OCA and $\text{MA}_{\aleph_1}$. Then every homeomorphism of $\omega^*$ is induced by a bijection $e : \omega \setminus F_1 \to \omega \setminus F_2$ where $F_1, F_2$ are finite.*
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Assume OCA. Then every automorphism of $B(\ell^2)/\mathcal{K}(\ell^2)$ is induced by a linear isometry between closed subspaces of $\ell^2$ with finite codimension.
Assume OCA and MA$_{\aleph_1}$. Let $X$ and $Y$ be zero-dimensional, locally compact, noncompact Polish spaces, and suppose $\varphi : C(X)/K(X) \to C(Y)/K(Y)$ is an isomorphism. Then there are compact sets $K \subseteq X$ and $L \subseteq Y$, and a homeomorphism $e : Y \setminus L \to X \setminus K$, such that $\varphi([A]) = [e^{-1}(A)]$ for all $A \in C(X)$. 

Proof. (Sketch, of a special case, using a big black box.) Let $X = \check{\bigcup}_{n=1}^{\infty} [T_n]$ where each $T_n$ is a finitely-branching tree. Given $s \in T_n$ we write $[s]$ for the set of $x \in [T_n]$ extending $s$. Let $P$ be the set of all partitions $Q$ of $X$ into sets of the form $[s]$. 

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Theorem

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Let $\mathcal{P}$ be the set of all partitions $Q$ of $X$ into sets of the form $[s]$. 
Given $Q \in \mathbb{P}$ there is a natural embedding

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$$\tau_{Q_2 Q_1} : \mathcal{P}(Q_2) / \text{fin} \rightarrow \mathcal{P}(Q_1) / \text{fin}$$

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**Lemma**

$\mathcal{C}(X)/\mathcal{K}(X)$ is the direct limit of the algebras $\mathcal{P}(Q)/\text{fin}$ along the connecting maps $\tau_{Q_2Q_1}$. 
For each $Q \in \mathbb{P}$, define $\varphi_Q = \varphi \circ \sigma_Q$.

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**Theorem (Veličković, essentially)**

Assume OCA and $MA_{\aleph_1}$. Then every embedding $\psi : \mathcal{P}(\omega)/\text{fin} \rightarrow \mathcal{C}(Y)/\mathcal{K}(Y)$ is of the form

$$\psi([A]) = [e^{-1}(A)]$$

for some continuous function $e : Y \rightarrow \omega$. 
For each $Q \in \mathbb{P}$, define $\varphi_Q = \varphi \circ \sigma_Q$.

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So we get continuous functions $e_Q : Y \rightarrow Q$ inducing the embeddings $\varphi_Q$. 
The fact that the $e_Q$'s induce the same isomorphism on different subalgebras implies that they are \textit{coherent} in the following way.
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Given $Q_1, Q_2$ define $Q_1 \vee Q_2$ to be the finest partition which is coarser than both $Q_1$ and $Q_2$. 
The fact that the $e_Q$’s induce the same isomorphism on different subalgebras implies that they are *coherent* in the following way.

Given $Q_1, Q_2$ define $Q_1 \vee Q_2$ to be the finest partition which is coarser than both $Q_1$ and $Q_2$.

Define $s_{Q_1, Q_1 \vee Q_2} : Q_1 \to Q_1 \vee Q_2$ by defining $s_{Q_1, Q_1 \vee Q_2}(U)$ to be the unique element $V$ of $Q_1 \vee Q_2$ such that $V \supseteq U$. 
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Then for any $Q_1, Q_2 \in \mathcal{P}$, the set

$$\Delta(e_{Q_1}, e_{Q_2}) = \{ y \in Y \mid s_{Q_1, Q_1 \vee Q_2}(e_{Q_1}(y)) \neq s_{Q_2, Q_1 \vee Q_2}(e_{Q_2}(y)) \}$$

is compact.
The fact that the $e_Q$'s induce the same isomorphism on different subalgebras implies that they are \textit{coherent} in the following way.

Given $Q_1, Q_2$ define $Q_1 \lor Q_2$ to be the finest partition which is coarser than both $Q_1$ and $Q_2$.

Define $s_{Q_1, Q_1 \lor Q_2} : Q_1 \to Q_1 \lor Q_2$ by defining $s_{Q_1, Q_1 \lor Q_2}(U)$ to be the unique element $V$ of $Q_1 \lor Q_2$ such that $V \supseteq U$.

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(We will say that $e_{Q_1}$ and $e_{Q_2}$ \textit{cohere exactly} if $\Delta(e_{Q_1}, e_{Q_2}) = \emptyset$.)
Define $G \subseteq [\mathbb{P}]^2$ to be the set of $\{Q_1, Q_2\}$ such that $e_{Q_1}$ and $e_{Q_2}$ do \textit{not} cohere exactly.
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- there is an uncountable $A \subseteq \mathcal{P}$ such that $[A]^2 \subseteq G$, or
- there is a partition $\mathcal{P} = \bigcup_n B_n$ such that for each $n$, $[B_n]^2 \cap G = \emptyset$. 
Assume $\text{MA}_{\aleph_1}$. Then there is no uncountable $A \subseteq \mathbb{P}$ such that $[A]^2 \subseteq G$. 

**Lemma**

Assume $\text{MA}_{\aleph_1}$. Then there is no uncountable $A \subseteq \mathbb{P}$ such that $[A]^2 \subseteq G$. 

**Sketch.**

WLOG $|A| = \aleph_1$. $\text{MA}_{\aleph_1}$ implies that there is some $Q \in \mathbb{P}$ which is $\preceq^*$ every $Q' \in A$. Then using the coherence of $e_Q$ with all of the $e_{Q'}$'s, along with a pigeonhole argument, we can find $Q_1, Q_2 \in A$ such that $e_{Q_1}$ and $e_{Q_2}$ cohere exactly.
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**Lemma**

If $\mathbb{P} = \bigcup_n B_n$ then one of the $B_n$'s is cofinal in $(\mathbb{P}, \preceq^*)$. 

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Lemma

If $P = \bigcup_n B_n$ then one of the $B_n$’s is cofinal in $(P, \prec^*)$.

Sketch.

$(P, \prec^*)$ is countably-directed.
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e is a homeomorphism from a cocompact subset of $Y$ to a cocompact subset of $X$, and for every $A \in \mathcal{C}(X)$,

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**Theorem (Gelfand Duality)**

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\( B(\ell^2)/\mathcal{K}(\ell^2) \) and \( C(X^*) \) are both C*-algebras of a special kind called corona algebras.
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Corollary

Assume $\text{OCA}_\infty$ and $\text{MA}_{\aleph_1}$. Let $X = \bigcup K_n$ and $Y = \bigcup L_n$ where $K_n$ and $L_n$ are compact Hausdorff spaces. Then every homeomorphism $X^* \simeq Y^*$ is induced by a sequence of homeomorphisms $K_n \simeq L_n$ (after possibly permuting the indices.)
Thank you!