Uniform Interpolation
Part II: An Algebraic Framework

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Theorem (Pitts 1992)

For any formula $\alpha(x, y)$ of intuitionistic propositional logic $\text{IL}$, there exist formulas $\alpha^L(y)$ and $\alpha^R(y)$ such that for any formula $\beta(y, z)$,

$$\alpha(x, y) \vdash_{\text{IL}} \beta(y, z) \iff \alpha^R(y) \vdash_{\text{IL}} \beta(y, z)$$

$$\beta(y, z) \vdash_{\text{IL}} \alpha(x, y) \iff \beta(y, z) \vdash_{\text{IL}} \alpha^L(y).$$

Other proofs of Pitts’ theorem have been given using **bisimulations** (Ghilardi 1995, Visser 1996) and **duality** (van Gool and Reggio 2018).
Remarks from Yesterday

- Other proofs of Pitts’ theorem have been given using *bisimulations* (Ghilardi 1995, Visser 1996) and *duality* (van Gool and Reggio 2018).

- There are exactly seven consistent *intermediate logics* that admit Craig interpolation (Maksimova 1977),
Remarks from Yesterday

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- There are exactly seven consistent intermediate logics that admit Craig interpolation (Maksimova 1977), and all of these also have uniform interpolation (Ghilardi and Zawadowski 2002).
Other proofs of Pitts’ theorem have been given using **bisimulations** (Ghilardi 1995, Visser 1996) and **duality** (van Gool and Reggio 2018).

There are exactly seven consistent **intermediate logics** that admit Craig interpolation (Maksimova 1977), and all of these also have uniform interpolation (Ghilardi and Zawadowski 2002).

Iemhoff has shown recently that any logic admitting a certain Dyckhoff-style decomposition calculus has uniform interpolation.
What does (uniform) interpolation mean algebraically?
Let $T_m(\overline{x})$ denote the term algebra of an algebraic language $\mathcal{L}$ with at least one constant over a set of variables $\overline{x}$. 
Equations and Equational Classes

Let $\textbf{T}_m(\bar{x})$ denote the term algebra of an algebraic language $\mathcal{L}$ with at least one constant over a set of variables $\bar{x}$.

An $\mathcal{L}$-equation is an ordered pair $\langle s, t \rangle$ of $\mathcal{L}$-terms, also written $s \approx t$.
Let $T_m(\bar{x})$ denote the term algebra of an algebraic language $\mathcal{L}$ with at least one constant over a set of variables $\bar{x}$.

An $\mathcal{L}$-equation is an ordered pair $\langle s, t \rangle$ of $\mathcal{L}$-terms, also written $s \approx t$.

We let $\mathcal{V}$ be any variety defined by $\mathcal{L}$-equations, e.g., Boolean algebras, Heyting algebras, MV-algebras, modal algebras, bounded lattices, groups...
For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ with variables in $\overline{x}$, we write

$$\Sigma \models_{\gamma} \varepsilon$$
For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ with variables in $\bar{x}$, we write

$$\Sigma \models_{\mathcal{V}} \varepsilon$$

if for any $A \in \mathcal{V}$ and homomorphism $e: \text{Tm}(\bar{x}) \to A$, where $\ker(e) := \{\langle s, t \rangle | e(s) = e(t)\}$ is the kernel of $e$. We also write $\Sigma \models_{\mathcal{V}} \Delta$ to denote that $\Sigma \models_{\mathcal{V}} \varepsilon$ for all $\varepsilon \in \Delta$. 
Equational Consequence

For any set of \( \mathcal{L} \)-equations \( \Sigma \cup \{\varepsilon\} \) with variables in \( \overline{x} \), we write

\[
\Sigma \models_{\mathcal{V}} \varepsilon
\]

if for any \( A \in \mathcal{V} \) and homomorphism \( e : Tm(\overline{x}) \to A \),

\[
\Sigma \subseteq \ker(e) \implies \varepsilon \in \ker(e),
\]

where \( \ker(e) := \{\langle s, t \rangle \mid e(s) = e(t)\} \) is the **kernel** of \( e \).
For any set of $\mathcal{L}$-equations $\Sigma \cup \{\varepsilon\}$ with variables in $\overline{x}$, we write

$$\Sigma \models_{\mathcal{V}} \varepsilon$$

if for any $A \in \mathcal{V}$ and homomorphism $e: Tm(\overline{x}) \to A$,

$$\Sigma \subseteq \text{ker}(e) \implies \varepsilon \in \text{ker}(e),$$

where $\text{ker}(e) := \{\langle s, t \rangle \mid e(s) = e(t)\}$ is the kernel of $e$.

We also write $\Sigma \models_{\mathcal{V}} \Delta$ to denote that $\Sigma \models_{\mathcal{V}} \varepsilon$ for all $\varepsilon \in \Delta$. 
\( \mathcal{V} \) admits **deductive interpolation** if whenever \( \Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \Delta(\overline{y}) \quad \text{and} \quad \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
\]
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\]

Equivalently, \( \mathcal{V} \) admits deductive interpolation if for any set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
\]
\( \mathcal{V} \) admits **deductive interpolation** if whenever \( \Sigma(\vec{x}, \vec{y}) \models \mathcal{V} \varepsilon(\vec{y}, \vec{z}) \), there exists a set of equations \( \Delta(\vec{y}) \) such that

\[
\Sigma(\vec{x}, \vec{y}) \models \mathcal{V} \Delta(\vec{y}) \quad \text{and} \quad \Delta(\vec{y}) \models \mathcal{V} \varepsilon(\vec{y}, \vec{z}).
\]

Equivalently, \( \mathcal{V} \) admits deductive interpolation if for any set of equations \( \Sigma(\vec{x}, \vec{y}) \), there exists a set of equations \( \Delta(\vec{y}) \) such that

\[
\Sigma(\vec{x}, \vec{y}) \models \mathcal{V} \varepsilon(\vec{y}, \vec{z}) \iff \Delta(\vec{y}) \models \mathcal{V} \varepsilon(\vec{y}, \vec{z}).
\]

(Just define \( \Delta(\vec{y}) \) := \{ \varepsilon(\vec{y}) \mid \Sigma(\vec{x}, \vec{y}) \models \mathcal{V} \varepsilon(\vec{y}) \}.)
A congruence $\Theta$ on an algebra $A$ is an equivalence relation satisfying

$$\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle \in \Theta \implies \langle \star(a_1, \ldots, a_n), \star(b_1, \ldots, b_n) \rangle \in \Theta$$

for every $n$-ary operation $\star$ of $A$. 
A congruence $\Theta$ on an algebra $A$ is an equivalence relation satisfying

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for every $n$-ary operation $\star$ of $A$.

The congruences of $A$ always form a complete lattice $\text{Con} A$. 
The **free algebra** of \( \mathcal{V} \) over a set of variables \( \overline{x} \) can be defined as

\[
F(\overline{x}) = Tm(\overline{x})/\Theta_{\mathcal{V}} \quad \text{where} \quad s \, \Theta_{\mathcal{V}} \, t \iff \mathcal{V} \models s \approx t.
\]
The **free algebra** of $\mathcal{V}$ over a set of variables $\bar{x}$ can be defined as

$$F(\bar{x}) = Tm(\bar{x})/\Theta_\mathcal{V} \quad \text{where} \quad s \Theta_\mathcal{V} t :\iff \mathcal{V} \models s \approx t.$$

We write $t$ to denote both a term $t$ in $Tm(\bar{x})$ and $[t]$ in $F(\bar{x})$. 
Lemma

For any set of equations $\Sigma \cup \{s \approx t\}$ with variables in $\overline{x}$, $\Sigma \models_V s \approx t \iff \langle s, t \rangle \in C_{g_{F(\overline{x})}}(\Sigma)$, where $C_{g_{F(\overline{x})}}(\Sigma)$ is the congruence on $F(\overline{x})$ generated by $\Sigma$. 
The inclusion map $i : F(\overline{y}) \to F(\overline{x}, \overline{y})$
The inclusion map $i : \mathbf{F}(\overline{y}) \to \mathbf{F}(\overline{x}, \overline{y})$ “lifts” to the maps

$$i^* : \operatorname{Con} \mathbf{F}(\overline{y}) \to \operatorname{Con} \mathbf{F}(\overline{x}, \overline{y}); \quad \Theta \mapsto \mathcal{C}_{\mathbf{F}(\overline{x}, \overline{y})}(i[\Theta])$$
The inclusion map $i : F(\bar{y}) \to F(\bar{x}, \bar{y})$ “lifts” to the maps

$$
\begin{align*}
\iota^* : \text{Con } F(\bar{y}) & \to \text{Con } F(\bar{x}, \bar{y}); & \Theta & \mapsto C_{g_{F(\bar{x}, \bar{y})}}(i[\Theta]) \\
i^{-1} : \text{Con } F(\bar{x}, \bar{y}) & \to \text{Con } F(\bar{y}); & \Psi & \mapsto i^{-1}[\Psi] = \Psi \cap F(\bar{y})^2.
\end{align*}
$$
The inclusion map $i: \mathbf{F}(\overline{y}) \to \mathbf{F}(\overline{x}, \overline{y})$ “lifts” to the maps

\[ i^*: \text{Con} \mathbf{F}(\overline{y}) \to \text{Con} \mathbf{F}(\overline{x}, \overline{y}); \quad \Theta \mapsto \text{Cg}_{\mathbf{F}(\overline{x}, \overline{y})}(i[\Theta]) \]

\[ i^{-1}: \text{Con} \mathbf{F}(\overline{x}, \overline{y}) \to \text{Con} \mathbf{F}(\overline{y}); \quad \Psi \mapsto i^{-1}[\Psi] = \Psi \cap \mathbf{F}(\overline{y})^2. \]

Note that the pair $\langle i^*, i^{-1} \rangle$ is an adjunction, i.e.,

\[ i^*(\Theta) \subseteq \Psi \iff \Theta \subseteq i^{-1}(\Psi). \]
The following are equivalent:

1. \( \forall \) admits **deductive interpolation**, i.e., for any set of equations \( \Sigma(x, y) \), there exists a set of equations \( \Delta(y) \) such that

\[
\Sigma(x, y) \models \forall \varepsilon(y, z) \iff \Delta(y) \models \forall \varepsilon(y, z).
\]
The following are equivalent:

1. \( \mathcal{V} \) admits **deductive interpolation**, i.e., for any set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models \mathcal{V} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models \mathcal{V} \varepsilon(\overline{y}, \overline{z}).
\]

2. For any finite sets \( \overline{x}, \overline{y}, \overline{z} \), the following diagram commutes:

\[
\begin{array}{ccc}
\text{Con } \mathbf{F}(\overline{x}, \overline{y}) & \xrightarrow{i^{-1}} & \text{Con } \mathbf{F}(\overline{y}) \\
\downarrow j^* & & \downarrow l^* \\
\text{Con } \mathbf{F}(\overline{x}, \overline{y}, \overline{z}) & \xrightarrow{k^{-1}} & \text{Con } \mathbf{F}(\overline{y}, \overline{z})
\end{array}
\]

where \( i, j, k, \) and \( l \) denote inclusion maps between free algebras.
\( \mathcal{V} \) admits the **amalgamation property** if for any \( A, B_1, B_2 \in \mathcal{V} \), and embeddings \( \sigma_1 : A \rightarrow B_1, \sigma_2 : A \rightarrow B_2 \),
$\mathcal{V}$ admits the **amalgamation property** if for any $A, B_1, B_2 \in \mathcal{V}$, and embeddings $\sigma_1 : A \rightarrow B_1$, $\sigma_2 : A \rightarrow B_2$, there exist $C \in \mathcal{V}$
The Amalgamation Property

\[ \mathcal{V} \text{ admits the \textbf{amalgamation property} if for any } A, B_1, B_2 \in \mathcal{V}, \text{ and embeddings } \sigma_1 : A \to B_1, \sigma_2 : A \to B_2, \text{ there exist } C \in \mathcal{V} \text{ and embeddings } \tau_1 : B_1 \to C \text{ and } \tau_2 : B_2 \to C \]
\(\mathcal{V}\) admits the **amalgamation property** if for any \(A, B_1, B_2 \in \mathcal{V}\), and embeddings \(\sigma_1 : A \rightarrow B_1\), \(\sigma_2 : A \rightarrow B_2\), there exist \(C \in \mathcal{V}\) and embeddings \(\tau_1 : B_1 \rightarrow C\) and \(\tau_2 : B_2 \rightarrow C\) such that \(\tau_1 \sigma_1 = \tau_2 \sigma_2\).
Lemma (Pigozzi 1972)

\( \mathcal{V} \) admits the amalgamation property if and only if for any disjoint sets \( \bar{x}, \bar{y}, \bar{z} \) and \( \Theta \in \text{Con } F(\bar{x}, \bar{y}) \), \( \Psi \in \text{Con } F(\bar{y}, \bar{z}) \) satisfying

\[ \Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2, \]

This property of congruences of free algebras can be reformulated in terms of consequence as the so-called Robinson property.
Lemma (Pigozzi 1972)

\( \mathcal{V} \) admits the amalgamation property if and only if for any disjoint sets \( \overline{x}, \overline{y}, \overline{z} \) and \( \Theta \in \text{Con} \ F(\overline{x}, \overline{y}), \ \Psi \in \text{Con} \ F(\overline{y}, \overline{z}) \) satisfying

\[
\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2,
\]

there exists \( \Phi \in \text{Con} \ F(\overline{x}, \overline{y}, \overline{z}) \) satisfying

\[
\Theta = \Phi \cap F(\overline{x}, \overline{y})^2 \quad \text{and} \quad \Psi = \Phi \cap F(\overline{y}, \overline{z})^2.
\]
A Key Lemma

Lemma (Pigozzi 1972)

\( \mathcal{V} \) admits the amalgamation property if and only if for any disjoint sets \( \overline{x}, \overline{y}, \overline{z} \) and \( \Theta \in \text{Con} \ F(\overline{x}, \overline{y}), \Psi \in \text{Con} \ F(\overline{y}, \overline{z}) \) satisfying

\[
\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2,
\]

there exists \( \Phi \in \text{Con} \ F(\overline{x}, \overline{y}, \overline{z}) \) satisfying

\[
\Theta = \Phi \cap F(\overline{x}, \overline{y})^2 \quad \text{and} \quad \Psi = \Phi \cap F(\overline{y}, \overline{z})^2.
\]

This property of congruences of free algebras can be reformulated in terms of consequence as the so-called **Robinson property**.
Suppose that \( \mathcal{V} \) admits the amalgamation property and \( \Theta \in \text{Con} \ F(\overline{x}, \overline{y}) \), \( \Psi \in \text{Con} \ F(\overline{y}, \overline{z}) \) satisfy \( \Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2 \).
Proof Sketch \((\Rightarrow)\)

Suppose that \(\mathcal{V}\) admits the amalgamation property and \(\Theta \in \text{Con} \, F(\overline{x}, \overline{y})\), \(\Psi \in \text{Con} \, F(\overline{y}, \overline{z})\) satisfy \(\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2\). We define

\[
A = F(\overline{y})/\Phi_0, \quad B = F(\overline{x}, \overline{y})/\Theta, \quad \text{and} \quad C = F(\overline{y}, \overline{z})/\Psi,
\]
Suppose that $\mathcal{V}$ admits the amalgamation property and $\Theta \in \text{Con } F(\bar{x}, \bar{y})$, $\Psi \in \text{Con } F(\bar{y}, \bar{z})$ satisfy $\Phi_0 := \Theta \cap F(\bar{y})^2 = \Psi \cap F(\bar{y})^2$. We define

$$A = F(\bar{y})/\Phi_0, \quad B = F(\bar{x}, \bar{y})/\Theta, \quad \text{and} \quad C = F(\bar{y}, \bar{z})/\Psi,$$

yielding an amalgam $D$.
Suppose that $\mathcal{V}$ admits the amalgamation property and $\Theta \in \text{Con} F(\overline{x}, \overline{y})$, $\Psi \in \text{Con} F(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$. We define

$$A = F(\overline{y})/\Phi_0, \quad B = F(\overline{x}, \overline{y})/\Theta, \quad \text{and} \quad C = F(\overline{y}, \overline{z})/\Psi,$$

yielding an amalgam $D$ and a surjective homomorphism $g : F(\overline{x}, \overline{y}, \overline{z}) \to D$.
Suppose that $\mathcal{V}$ admits the amalgamation property and $\Theta \in \text{Con} \, F(\overline{x}, \overline{y})$, $\Psi \in \text{Con} \, F(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$. We define

$$A = F(\overline{y})/\Phi_0, \quad B = F(\overline{x}, \overline{y})/\Theta, \quad \text{and} \quad C = F(\overline{y}, \overline{z})/\Psi,$$

yielding an amalgam $D$ and a surjective homomorphism $g : F(\overline{x}, \overline{y}, \overline{z}) \to D$ with $\Phi := \ker(g)$ satisfying $\Theta = \Phi \cap F(\overline{x}, \overline{y})^2$ and $\Psi = \Phi \cap F(\overline{y}, \overline{z})^2$. 

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**Proof Sketch ($\Rightarrow$)**

Let $\mathcal{V}$ be a relation. Suppose that $\mathcal{V}$ admits the amalgamation property and $\Theta \in \text{Con} \, F(\overline{x}, \overline{y})$, $\Psi \in \text{Con} \, F(\overline{y}, \overline{z})$ satisfy $\Phi_0 := \Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$. We define

$$A = F(\overline{y})/\Phi_0, \quad B = F(\overline{x}, \overline{y})/\Theta, \quad \text{and} \quad C = F(\overline{y}, \overline{z})/\Psi,$$

yielding an amalgam $D$ and a surjective homomorphism $g : F(\overline{x}, \overline{y}, \overline{z}) \to D$ with $\Phi := \ker(g)$ satisfying $\Theta = \Phi \cap F(\overline{x}, \overline{y})^2$ and $\Psi = \Phi \cap F(\overline{y}, \overline{z})^2$. 

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**Diagram:**

```
F(x, y) ⊆ F(x, y, z)
  |   |   |
  |   |   g
  |   |   |
F(y) ⊆ F(y, z) ⊆ D
<--|        |
    |        |
    |        |
A   |        D
    |        |
    |        |
    |        |
    B       |
    |        |
    |        |
    |        |
    |        |
    |        |
    |        |
    |        |
    C       |
```
Proof Sketch ($\iff$)

Let $B, C \in \mathcal{V}$ be generated by $\overline{x}, \overline{y}$ and $\overline{y}, \overline{z}$, respectively, with a common subalgebra $A$ generated by $\overline{y}$. 

Consider the surjective homomorphisms $\pi_A: F(x, y) \to A$, $\pi_B: F(x, y) \to B$, and $\pi_C: F(y, z) \to C$.

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(V(y))^2 = \Psi \cap F(y)^2$ so, by assumption, there exists $\Phi \in \text{Con} F(x, y, z)$ such that $\Phi \cap F(x, y)^2 = \Theta$ and $\Phi \cap F(y, z)^2 = \Psi$.

The required amalgam is then $D = F(x, y, y) / \Phi$.
Let $B, C \in \mathcal{V}$ be generated by $\overline{x}, \overline{y}$ and $\overline{y}, \overline{z}$, respectively, with a common subalgebra $A$ generated by $\overline{y}$. Consider the surjective homomorphisms

$$
\pi_A : F(\overline{y}) \to A, \quad \pi_B : F(\overline{x}, \overline{y}) \to B, \quad \text{and} \quad \pi_C : F(\overline{y}, \overline{z}) \to C.
$$

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(V(\overline{y}))^2 = \Psi \cap F(\overline{y})^2$, so, by assumption, there exists $\Phi \in \text{Con} F(\overline{x}, \overline{y}, \overline{z})$ such that $\Phi \cap F(\overline{x}, \overline{y})^2 = \Theta$ and $\Phi \cap F(\overline{y}, \overline{z})^2 = \Psi$.

The required amalgam is then $D = F(\overline{x}, \overline{y}, \overline{y}) / \Phi$. 

![Diagram](image-url)
Let $B, C \in \mathcal{V}$ be generated by $\mathbf{x}, \mathbf{y}$ and $\mathbf{y}, \mathbf{z}$, respectively, with a common subalgebra $A$ generated by $\mathbf{y}$. Consider the surjective homomorphisms

$$
\pi_A : F(\mathbf{y}) \to A, \quad \pi_B : F(\mathbf{x}, \mathbf{y}) \to B, \quad \text{and} \quad \pi_C : F(\mathbf{y}, \mathbf{z}) \to C.
$$

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F_{\mathcal{V}}(\mathbf{y})^2 = \Psi \cap F(\mathbf{y})^2$.

The required amalgam is then $D = F(\mathbf{x}, \mathbf{y}, \mathbf{y}) / \Phi$.
Proof Sketch ($\iff$)

Let $B, C \in \mathcal{V}$ be generated by $\overline{x}, \overline{y}$ and $\overline{y}, \overline{z}$, respectively, with a common subalgebra $A$ generated by $\overline{y}$. Consider the surjective homomorphisms

$$
\pi_A : F(\overline{y}) \to A, \quad \pi_B : F(\overline{x}, \overline{y}) \to B, \quad \text{and} \quad \pi_C : F(\overline{y}, \overline{z}) \to C.
$$

Then $\Theta = \ker(\pi_B)$, $\Psi = \ker(\pi_C)$ satisfy $\Theta \cap F(\overline{y})^2 = \Psi \cap F(\overline{y})^2$ so, by assumption, there exists $\Phi \in \text{Con } F(\overline{x}, \overline{y}, \overline{z})$ such that $\Phi \cap F(\overline{x}, \overline{y})^2 = \Theta$ and $\Phi \cap F(\overline{y}, \overline{z})^2 = \Psi$. 

The required amalgam is then $D = F(\overline{x}, \overline{y}, \overline{z}) / \Phi$. 

\[\begin{array}{ccc}
F(\overline{x}, \overline{y}) & \hookrightarrow & F(\overline{x}, \overline{y}, \overline{z}) \\
| & & |
\pi_B \\
F(\overline{y}) & \hookrightarrow & F(\overline{y}, \overline{z}) \\
\downarrow & & \downarrow \\
B & \hookrightarrow & D \\
\pi_A \\
| & & |
A & \hookrightarrow & C \\
\downarrow & & \downarrow \\
\pi_B & & \pi_C
\end{array}\]
Proof Sketch \( \langle \Rightarrow \rangle \)

Let \( B, C \in \mathcal{V} \) be generated by \( \bar{x}, \bar{y} \) and \( \bar{y}, \bar{z} \), respectively, with a common subalgebra \( A \) generated by \( \bar{y} \). Consider the surjective homomorphisms

\[
\pi_A : F(\bar{y}) \to A, \quad \pi_B : F(\bar{x}, \bar{y}) \to B, \quad \text{and} \quad \pi_C : F(\bar{y}, \bar{z}) \to C.
\]

Then \( \Theta = \ker(\pi_B), \Psi = \ker(\pi_C) \) satisfy \( \Theta \cap F_\mathcal{V}(\bar{y})^2 = \Psi \cap F(\bar{y})^2 \) so, by assumption, there exists \( \Phi \in \text{Con} F(\bar{x}, \bar{y}, \bar{z}) \) such that \( \Phi \cap F(\bar{x}, \bar{y})^2 = \Theta \) and \( \Phi \cap F(\bar{y}, \bar{z})^2 = \Psi \). The required amalgam is then \( D = F(\bar{x}, \bar{y}, \bar{y})/\Phi \).

\[\begin{align*}
F(\bar{x}, \bar{y}) & \xrightarrow{\pi_A} A \\
F(\bar{y}) & \xrightarrow{\pi_B} B \\
F(\bar{y}, \bar{z}) & \xrightarrow{\pi_C} C \\
F(\bar{x}, \bar{y}, \bar{y})/\Phi & \xrightarrow{\text{amalgam}} D
\end{align*}\]
A Bridge Theorem

Theorem (Pigozzi, Bacsich, Maksimova, Czelakowski, …)

A variety with the congruence extension property admits the deductive interpolation property if and only if it admits the amalgamation property.
Further Relationships.

$$\text{CEP} + \text{FAP}$$

$$\iff$$

$$\text{TIP} \implies \text{AP} \implies \text{WAP} \implies \text{FAP}$$

$$\iff$$

$$\text{MIP} \implies \text{RP} \implies \text{CDIP} \implies \text{DIP}$$

$$\iff$$

$$\text{DIP} + \text{EP}$$

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G. Metcalfe, F. Montagna, and C. Tsinakis.
Amalgamation and interpolation in ordered algebras.
Can we describe **uniform interpolation** algebraically?
$\mathcal{V}$ has **deductive interpolation** if for any set of equations $\Sigma(x, y)$, there exists a set of equations $\Delta(y)$ such that

$$
\Sigma(x, y) \models_{\mathcal{V}} \varepsilon(y, z) \iff \Delta(y) \models_{\mathcal{V}} \varepsilon(y, z).
$$

Equivalently, $\mathcal{V}$ has deductive interpolation and for any finite set of equations $\Sigma(x, y)$, there exists a finite set of equations $\Delta(y)$ such that

$$
\Sigma(x, y) \models_{\mathcal{V}} \varepsilon(y) \iff \Delta(y) \models_{\mathcal{V}} \varepsilon(y).
$$

But what does this extra ingredient mean algebraically?
\( \mathcal{V} \) has **right uniform deductive interpolation** if for any *finite* set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a *finite* set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
\]
$\mathcal{V}$ has **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Delta(\overline{y})$ such that

$$\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).$$

Equivalently, $\mathcal{V}$ has deductive interpolation and for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that

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\]

But what does this extra ingredient mean *algebraically*?
The **compact** (equivalently, finitely generated) congruences of an algebra $A$ always form a join-semilattice $KCon A$. 
The **compact** (equivalently, finitely generated) congruences of an algebra $A$ always form a join-semilattice $KCon_A$.

Recall that the inclusion map $i: F(\bar{y}) \to F(\bar{x}, \bar{y})$ “lifts” to the maps

$$i^* : \text{Con } F(\bar{y}) \to \text{Con } F(\bar{x}, \bar{y}); \quad \Theta \mapsto C_{g_F(\bar{x}, \bar{y})}(i[\Theta])$$

$$i^{-1} : \text{Con } F(\bar{x}, \bar{y}) \to \text{Con } F(\bar{y}); \quad \psi \mapsto i^{-1}[\psi] = \psi \cap F(\bar{y})^2.$$
Compact Congruences

The **compact** (equivalently, finitely generated) congruences of an algebra \( A \) always form a join-semilattice \( K\text{Con} A \).

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The **compact lifting** of \( i \) restricts \( i^* \) to \( K\text{Con} F(\bar{y}) \to K\text{Con} F(\bar{x}, \bar{y}); \)
Compact Congruences

The compact (equivalently, finitely generated) congruences of an algebra \( A \) always form a join-semilattice \( K\text{Con} A \).

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\]

The compact lifting of \( i \) restricts \( i^* \) to \( K\text{Con} F(\bar{y}) \to K\text{Con} F(\bar{x}, \bar{y}) \); it has a right adjoint if \( i^{-1} \) restricts to \( K\text{Con} F(\bar{x}, \bar{y}) \to K\text{Con} F(\bar{y}) \).
An algebra $A \in \mathcal{V}$ is called

- **finitely generated** if it is generated by a finite subset of $A$;

- **finitely presented** if it is isomorphic to $F(x)/\Theta$ for some finite set $x$ and compact congruence $\Theta$ on $F(x)$.
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**Useful Lemma**

*If $A \in \mathcal{V}$ is finitely presented and isomorphic to $F(\bar{x})/\Theta$ for some finite set $\bar{x}$ and congruence $\Theta$ on $F(\bar{x})$, then $\Theta$ is compact.*
The notion of coherence originated in sheaf theory and has been studied quite widely in algebra, e.g., in connection with groups, rings, and monoids.
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Following Wheeler, a variety $\mathcal{V}$ is **coherent** if all finitely generated subalgebras of finitely presented members of $\mathcal{V}$ are finitely presented.


Note that clearly every locally finite variety is coherent. (Homework question. Is your favourite variety coherent?)
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The Missing Ingredient

Theorem (Kowalski and Metcalfe 2017)

The following are equivalent:

1. For any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there is a finite set of equations $\Delta(\overline{y})$ such that

$$
\Sigma(\overline{x}, \overline{y}) \models \nu \varepsilon(\overline{y}) \iff \Delta(\overline{y}) \models \nu \varepsilon(\overline{y}).
$$

2. For finite $\overline{x}, \overline{y}$, the compact lifting of $F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y})$ has a right adjoint; that is, $\Theta \in \mathcal{KCon}_{F(\overline{x}, \overline{y})} \Rightarrow \Theta \cap F(\overline{y}) \in \mathcal{KCon}_{F(\overline{y})}$.

3. $\nu$ is coherent.

Corollary (Pitts 1992)

The variety of Heyting algebras is coherent.
Theorem (Kowalski and Metcalfe 2017)

The following are equivalent:

(1) For any finite set of equations $\Sigma(x, y)$, there is a finite set of equations $\Delta(y)$ such that

$$\Sigma(x, y) \models \varepsilon(y) \iff \Delta(y) \models \varepsilon(y).$$

(2) For finite $x, y$, the compact lifting of $F(y) \hookrightarrow F(x, y)$ has a right adjoint; that is, $\Theta \in KCon F(x, y) \implies \Theta \cap F(y)^2 \in KCon F(y)$. 

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The variety of Heyting algebras is coherent.
The Missing Ingredient

Theorem (Kowalski and Metcalfe 2017)

The following are equivalent:

(1) For any finite set of equations $\Sigma(x, y)$, there is a finite set of equations $\Delta(y)$ such that

$$\Sigma(x, y) \models \vartheta(y) \iff \Delta(y) \models \vartheta(y).$$

(2) For finite $x, y$, the compact lifting of $F(y) \hookrightarrow F(x, y)$ has a right adjoint; that is, $\Theta \in \text{KCon } F(x, y) \implies \Theta \cap F(y)^2 \in \text{KCon } F(y)$.

(3) $\vartheta$ is coherent.

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The variety of Heyting algebras is coherent.
Proof of \((2) \Leftrightarrow (3)\)

**Proof.**

We prove that \(\mathcal{V}\) is coherent if and only if for any finite \(\bar{x}, \bar{y}\),

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\Theta \in \text{KCon } F(\bar{x}, \bar{y}) \iff \Theta \cap F(\bar{y})^2 \in \text{KCon } F(\bar{y}).
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$(\implies)$ Let $\mathcal{V}$ be coherent and consider finite $\bar{x}$, $\bar{y}$ and $\Theta \in \text{KCon } F(\bar{x}, \bar{y})$. 

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Let $\mathcal{V}$ be coherent and consider finite $\bar{x}$, $\bar{y}$ and $\Theta \in \text{KCon } F(\bar{x}, \bar{y})$. 

Hence $B$ is finitely presented. 

George Metcalfe (University of Bern) 
Uniform Interpolation 
August 2018 25 / 30
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(⇒) Let \( \mathcal{V} \) be coherent and consider finite \( \overline{x}, \overline{y} \) and \( \Theta \in \text{KCon } F(\overline{x}, \overline{y}) \). \( F(\overline{x}, \overline{y})/\Theta \) is finitely presented and, by coherence, so is \( F(\overline{y})/(\Theta \cap F(\overline{y})^2) \).
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\((\Rightarrow)\) Let \(\mathcal{V}\) be coherent and consider finite \(\bar{x}, \bar{y}\) and \(\Theta \in \text{KCon } \mathbf{F}(\bar{x}, \bar{y})\). \(\mathbf{F}(\bar{x}, \bar{y})/\Theta\) is finitely presented and, by coherence, so is \(\mathbf{F}(\bar{y})/(\Theta \cap \mathbf{F}(\bar{y})^2)\). Hence, by the useful lemma, \(\Theta \cap \mathbf{F}(\bar{y})^2 \in \text{KCon } \mathbf{F}(\bar{y})\).

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We prove that $\mathcal{V}$ is coherent if and only if for any finite $\overline{x}$, $\overline{y}$,

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$(\Leftarrow)$ Let $\mathcal{B}$ be a finitely generated subalgebra of a finitely presented $A \in \mathcal{V}$. Let $\overline{x}$, $\overline{y}$ and $\overline{y}$ be finite sets generating $A$ and $\mathcal{B}$, respectively.
Proof of \((2) \iff (3)\)

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\((\impliedby)\) Let \(\mathcal{B}\) be a finitely generated subalgebra of a finitely presented \(\mathcal{A} \in \mathcal{V}\). Let \(\bar{x}, \bar{y}\) and \(\bar{y}\) be finite sets generating \(\mathcal{A}\) and \(\mathcal{B}\), respectively. The onto homomorphism \(h: F(\bar{x}, \bar{y}) \to \mathcal{A}\) restricts to \(k: F(\bar{y}) \to \mathcal{B}\), which is also onto.
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\((\Leftarrow)\) Let \(\mathcal{B}\) be a finitely generated subalgebra of a finitely presented \(\mathcal{A} \in \mathcal{V}\). Let \(\bar{x}, \bar{y}\) and \(\bar{y}\) be finite sets generating \(\mathcal{A}\) and \(\mathcal{B}\), respectively. The onto homomorphism \(h: F(\bar{x}, \bar{y}) \to \mathcal{A}\) restricts to \(k: F(\bar{y}) \to \mathcal{B}\), which is also onto. But \(\ker h \in \text{KCon } F(\bar{x}, \bar{y})\) by the useful lemma,
Proof of (2) ⇔ (3)

Proof.

We prove that $\mathcal{V}$ is coherent if and only if for any finite $\bar{x}$, $\bar{y}$,

$$\Theta \in K\text{Con} F(\bar{x}, \bar{y}) \implies \Theta \cap F(\bar{y})^2 \in K\text{Con} F(\bar{y}).$$

($\implies$) Let $\mathcal{V}$ be coherent and consider finite $\bar{x}$, $\bar{y}$ and $\Theta \in K\text{Con} F(\bar{x}, \bar{y})$. $F(\bar{x}, \bar{y})/\Theta$ is finitely presented and, by coherence, so is $F(\bar{y})/(\Theta \cap F(\bar{y})^2)$. Hence, by the useful lemma, $\Theta \cap F(\bar{y})^2 \in K\text{Con} F(\bar{y})$.

($\impliedby$) Let $\mathcal{B}$ be a finitely generated subalgebra of a finitely presented $A \in \mathcal{V}$. Let $\bar{x}$, $\bar{y}$ and $\bar{y}$ be finite sets generating $A$ and $B$, respectively. The onto homomorphism $h: F(\bar{x}, \bar{y}) \to A$ restricts to $k: F(\bar{y}) \to B$, which is also onto. But $\ker h \in K\text{Con} F(\bar{x}, \bar{y})$ by the useful lemma, so by assumption, $\ker k = \ker h \cap F(\bar{y})^2 \in K\text{Con} F(\bar{y})$. 
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Another Bridge Theorem

Theorem (Kowalski and Metcalfe 2017)
A variety with the congruence extension property has right uniform deductive interpolation if and only if it is coherent and admits the amalgamation property.
\( \mathcal{V} \) has **left uniform deductive interpolation** if for any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a finite set of equations \( \Delta(\overline{y}) \) such that

\[
\Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Sigma(\overline{x}, \overline{y}) \iff \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Delta(\overline{y}).
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\( \mathcal{V} \) has left uniform deductive interpolation if for any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a finite set of equations \( \Delta(\overline{y}) \) such that

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\]

**Lemma**

The following are equivalent:

1. \( \mathcal{V} \) has left uniform deductive interpolation.
2. \( \mathcal{V} \) has deductive interpolation, and for finite sets \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) has a left adjoint.
   
   Moreover, if \( \mathcal{V} \) is locally finite, these are equivalent to
   
   3. \( \mathcal{V} \) has deductive interpolation, is congruence distributive, and for finite sets \( \overline{x}, \overline{y} \), the compact lifting of \( F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y}) \) preserves meets.
$\mathcal{V}$ has **left uniform deductive interpolation** if for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that

$$\Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Sigma(\overline{x}, \overline{y}) \iff \Pi(\overline{y}, \overline{z}) \models_{\mathcal{V}} \Delta(\overline{y}).$$

**Lemma**

The following are equivalent:

1. $\mathcal{V}$ has **left uniform deductive interpolation**.
2. $\mathcal{V}$ has deductive interpolation, and for finite sets $\overline{x}, \overline{y}$, the compact lifting of $F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y})$ has a left adjoint.

Moreover, if $\mathcal{V}$ is locally finite, these are equivalent to

3. $\mathcal{V}$ has deductive interpolation, is congruence distributive, and for finite sets $\overline{x}, \overline{y}$, the compact lifting of $F(\overline{y}) \hookrightarrow F(\overline{x}, \overline{y})$ preserves meets.
A first-order theory $T^*$ is a **model completion** of a universal theory $T$ if

(a) $T$ and $T^*$ entail the same universal sentences;

(b) $T^*$ admits quantifier elimination.

Moreover, $T^*$ is then the theory of the existentially closed models for $T$.
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Moreover, $T^*$ is then the theory of the existentially closed models for $T$.

**Theorem (Wheeler 1976)**

The theory of $\mathcal{V}$ has a model completion if and only if $\mathcal{V}$ is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.
A General Theorem

Theorem (Ghilardi and Zawadowski 2002, van Gool et al. 2017)

Suppose that
(i) $V$ is coherent and has the amalgamation property;
(ii) For finite sets $x, y$, the compact lifting of $F(y) \hookrightarrow F(x, y)$ has a left adjoint, and $KCon F(x)$ is dually Brouwerian.

Then the theory of $V$ has a model completion.

Corollary (Ghilardi and Zawadowski 1997)
The theory of Heyting algebras has a model completion.

Uniform interpolation and compact congruences.
A General Theorem

Theorem (Ghilardi and Zawadowski 2002, van Gool et al. 2017)

Suppose that

(i) $\mathcal{V}$ is coherent and has the amalgamation property;

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Uniform interpolation and compact congruences.
**Theorem (Ghilardi and Zawadowski 2002, van Gool et al. 2017)**

Suppose that

(i) \( \mathcal{V} \) is coherent and has the amalgamation property;

(ii) For finite sets \( \vec{x}, \vec{y} \), the compact lifting of \( F(\vec{y}) \hookrightarrow F(\vec{x}, \vec{y}) \) has a left adjoint, and \( K\text{Con} F(\vec{x}) \) is dually Brouwerian.

---

Uniform interpolation and compact congruences.  
Theorem (Ghilardi and Zawadowski 2002, van Gool et al. 2017)

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(i) $\mathcal{V}$ is coherent and has the amalgamation property;

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Uniform interpolation and compact congruences.
A General Theorem

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Then the theory of \( \mathcal{V} \) has a model completion.

Corollary (Ghilardi and Zawadowski 1997)

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Uniform interpolation and compact congruences.
We investigate uniform interpolation for some particular case studies, including varieties of modal algebras, lattices, and residuated lattices.
Tomorrow

- We investigate uniform interpolation for some particular case studies, including varieties of modal algebras, lattices, and residuated lattices.

- We provide a general criterion for establishing the failure of coherence and hence also of uniform interpolation.
We investigate uniform interpolation for some particular case studies, including varieties of modal algebras, lattices, and residuated lattices.

We provide a general criterion for establishing the failure of coherence and hence also of uniform interpolation.

We pose some open problems and challenges...