Uniform Interpolation
Part 3: Case Studies

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University of Bern

BLAST 2018, University of Denver, 6-10 August 2018
Yesterday... 

- we described a general algebraic framework for (uniform) interpolation in varieties of algebras and connections with properties such as amalgamation, coherence, and existence of a model completion.
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Today...  

- we consider case studies and general criteria for uniform interpolation, focussing first on varieties of algebras for modal logics.
**Modal logics** are used to reason about modal notions such as necessity, knowledge, obligation, and proof; they correspond to expressive but computationally tractable fragments of first-order logic.
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Description logics are multi-modal logics for reasoning about concept descriptions built from atomic concepts and roles such as

\[ \text{Woman} \mathbf{□} \forall \text{child}. \text{Woman} \quad \text{“women having only daughters”} \]
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However, we consider here only the basic language of classical logic extended with a unary connective $\Box$, defining also $\Diamond \alpha := \neg \Box \neg \alpha$. 
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Modal logics may be presented syntactically via axiom systems, sequent calculi, etc., and semantically via Kripke models, modal algebras, etc.
A Kripke frame $\langle W, R \rangle$ is an ordered pair consisting of a non-empty set of worlds $W$ and a binary accessibility relation $R \subseteq W \times W$. 
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A **Kripke model** \( \mathcal{M} = \langle W, R, \models \rangle \) consists of a Kripke frame \( \langle W, R \rangle \) together with a binary relation \( \models \) between worlds and formulas satisfying:

- \( w \models \alpha \land \beta \) if and only if \( w \models \alpha \) and \( w \models \beta \).
- \( w \models \alpha \lor \beta \) if and only if \( w \models \alpha \) or \( w \models \beta \).
- \( w \models \neg \alpha \) if and only if \( w \not\models \alpha \).
- \( w \models \Box \alpha \) if and only if for all \( v \in W \) such that \( Rwv \).

A formula \( \alpha \) is **valid** in \( \mathcal{M} \), written \( \mathcal{M} \models \alpha \), if \( w \models \alpha \) for all \( w \in W \).
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The basic modal logic K can be defined by extending any axiomatization of classical propositional logic with the axiom schema

\[(K) \quad \Box(\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)\]

and the *necessitation rule*: from \(\alpha\), infer \(\Box \alpha\).
Normal Modal Logics

The basic modal logic K can be defined by extending any axiomatization of classical propositional logic with the axiom schema

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Completeness

A normal modal logic $L$ is said to be complete with respect to a class of frames $C$. The following normal modal logics are complete with respect to the given class of frames:

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Moreover, all these normal modal logics have the finite model property.
A normal modal logic $L$ is said to be **complete** with respect to a class of frames $C$ if for any formula $\alpha$,

$$\vdash_L \alpha \iff M \models \alpha$$

for every model $M$ based on a frame in $C$. 

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A modal algebra consists of a Boolean algebra supplemented with a unary operation $\Box$ satisfying

$$\Box(x \land y) \approx \Box x \land \Box y \quad \text{and} \quad \Box \top \approx \top.$$
A **modal algebra** consists of a Boolean algebra supplemented with a unary operation □ satisfying

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\]

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A modal algebra consists of a Boolean algebra supplemented with a unary operation $\Box$ satisfying

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We let $\mathcal{K}$ denote the variety of all modal algebras.

In particular, each Kripke frame $\langle W, R \rangle$ yields a complex modal algebra

$$\langle \mathcal{P}(W), \cap, \cup, ^c, \emptyset, W, \Box \rangle \quad \text{where} \quad \Box A := \{ w \in W \mid Rwv \text{ for all } v \in A \}.$$
Theorem

For any normal modal logic $L$, let

$$\mathcal{V}_L = \{ A \in \mathcal{K} \mid \vdash_L \alpha \Rightarrow A \models \alpha \approx T \}.$$
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Then $\mathcal{V}_L$ is an **equivalent algebraic semantics** for $L$.
Theorem

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Then $V_L$ is an equivalent algebraic semantics for $L$ with transformers

$$\tau(\alpha) = \alpha \approx \top \quad \text{and} \quad \rho(\alpha \approx \beta) = (\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha).$$
For any normal modal logic $L$, let

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\mathcal{V}_L = \{ A \in \mathcal{K} \mid \vdash_L \alpha \implies A \models \alpha \approx T \}.
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\tau(\alpha) = \alpha \approx T \quad \text{and} \quad \rho(\alpha \approx \beta) = (\alpha \to \beta) \land (\beta \to \alpha).
$$

That is, for any set of formulas $T \cup \{ \alpha, \beta \}$ and set of equations $\Sigma$,

(a) $T \vdash_L \alpha \iff \tau[T] \models_{\mathcal{V}_L} \tau(\alpha)$;
(b) $\Sigma \models_{\mathcal{V}_L} \alpha \approx \beta \iff \rho[T] \vdash_L \rho(\alpha \approx \beta)$;
(c) $\alpha \vdash_L \rho(\tau(\alpha)) \quad \text{and} \quad \rho(\tau(\alpha)) \vdash_L \alpha$;
(d) $\alpha \approx \beta \models_{\mathcal{V}_L} \tau(\rho(\alpha \approx \beta)) \quad \text{and} \quad \tau(\rho(\alpha \approx \beta)) \models_{\mathcal{V}_L} \alpha \approx \beta$. 

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A normal modal logic \( L \) admits **deductive interpolation**
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$$\alpha(x, y) \vdash_L \beta(y, z) \quad \implies \quad \alpha \vdash_L \gamma \text{ and } \gamma \vdash_L \beta \quad \text{for some } \gamma(y)$$

if and only if $V_L$ admits the **amalgamation property**.
A normal modal logic $L$ admits **deductive interpolation**

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for some $\gamma(y)$ if and only if $\mathcal{V}_L$ admits the **amalgamation property**.

For example, $K$, $K_4$, $S_4$, $GL$, and somewhere between 43 and 49 axiomatic extensions of $S_4$ admit deductive interpolation, but not $S_5$.

Note, however, that $L$ admits **Craig interpolation**

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Theorem (Ghilardi 1995, Visser 1996, Bílková 2007)

$K$ has uniform interpolation.
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*K* has uniform interpolation.

Theorem (Kowalski and Metcalfe 2017)

*K* does not have uniform interpolation.
Theorem (Ghilardi 1995, Visser 1996, Bílková 2007)

*K has uniform Craig interpolation*

Theorem (Kowalski and Metcalfe 2017)

*K does not have uniform deductive interpolation.*
Theorem (Ghilardi 1995, Visser 1996, Bílková 2007)

*K* has uniform **Craig** interpolation; that is, for any formula $\alpha(x, y)$, there exist formulas $\alpha^L(y)$ and $\alpha^R(y)$ such that

\[
\vdash_K \alpha(x, y) \rightarrow \beta(y, z) \iff \vdash_K \alpha^R(y) \rightarrow \beta(y, z)
\]

\[
\vdash_K \beta(y, z) \rightarrow \alpha(x, y) \iff \vdash_K \beta(y, z) \rightarrow \alpha^L(y).
\]

Theorem (Kowalski and Metcalfe 2017)

*K* does not have uniform **deductive** interpolation.
$\mathcal{V}$ has **deductive interpolation** if for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Delta(\overline{y})$ such that

$$
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
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$\forall$ has **right uniform deductive interpolation** if for any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Delta(\overline{y})$ such that

$\Sigma(\overline{x}, \overline{y}) \models \forall \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models \forall \varepsilon(\overline{y}, \overline{z})$.
Recall... 

\( \mathcal{V} \) has **right uniform deductive interpolation** if for any *finite* set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a *finite* set of equations \( \Delta(\overline{y}) \) such that

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\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}, \overline{z}).
\]

Equivalently, \( \mathcal{V} \) has deductive interpolation and for any finite set of equations \( \Sigma(\overline{x}, \overline{y}) \), there exists a finite set of equations \( \Delta(\overline{y}) \) such that

\[
\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}) \iff \Delta(\overline{y}) \models_{\mathcal{V}} \varepsilon(\overline{y}).
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Recall also...

**Theorem (Kowalski and Metcalfe 2017)**

The following are equivalent:

1. For any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Delta(\overline{y})$ such that
   
   $$\Sigma(\overline{x}, \overline{y}) \models \nu \varepsilon(\overline{y}) \iff \Delta(\overline{y}) \models \nu \varepsilon(\overline{y}).$$

2. For finite $\overline{x}$, $\overline{y}$, the compact lifting of $\mathcal{F}(\overline{y}) \hookrightarrow \mathcal{F}(\overline{x}, \overline{y})$ has a right adjoint; that is, $\Theta \in \text{KCon} \mathcal{F}(\overline{x}, \overline{y}) = \Rightarrow \Theta \cap \mathcal{F}(\overline{y})^2 \in \text{KCon} \mathcal{F}(\overline{y})$.

3. $\nu$ is coherent, i.e., all finitely generated subalgebras of finitely presented members of $\nu$ are finitely presented.
Recall also...

**Theorem (Kowalski and Metcalfe 2017)**

The following are equivalent:

1. For any finite set of equations $\Sigma(x, y)$, there exists a finite set of equations $\Delta(y)$ such that
   $$\Sigma(x, y) \models \varepsilon(y) \iff \Delta(y) \models \varepsilon(y).$$

2. For finite $x, y$, the compact lifting of $F(y) \hookrightarrow F(x, y)$ has a right adjoint; that is, $\Theta \in K\text{Con} F(x, y) \implies \Theta \cap F(y)^2 \in K\text{Con} F(y)$.

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The following are equivalent:

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\Sigma(x, y) \models_v \varepsilon(y) \iff \Delta(y) \models_v \varepsilon(y).
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Theorem (Kowalski and Metcalfe 2017)

The variety of modal algebras is not coherent; so it does not admit uniform deductive interpolation and its theory does not have a model completion.

T. Kowalski and G. Metcalfe.

T. Kowalski and G. Metcalfe.
Proof

Let $\Box \alpha := \Box \alpha \land \alpha$, 

Claim. $\Sigma | = K \varepsilon (y, z) \iff \Delta | = K \varepsilon (y, z)$. 

It follows that if $K$ were coherent, then $\Delta | = K \Delta$ for some finite $\Delta' \subseteq \Delta$, and from this that $K | = \Box n z \approx \Box n + 1 z$ for some $n \in \mathbb{N}$, a contradiction.

Proof of claim. ($\Leftarrow$) Just observe that $\Sigma | = K \Delta$.

($\Rightarrow$) Suppose that $\Delta \not\subseteq K \varepsilon (y, z)$. Then there is a complete modal algebra $A$ and homomorphism $e : Tm(y, z) \to A$ such that $\Delta \subseteq \ker(e)$ and $\varepsilon \not\in \ker(e)$. Extend $e$ with $e(x) = \bigwedge_{k \in \mathbb{N}} \Box k e(z)$.

Then also $\Sigma \subseteq \ker(e)$, and hence $\Sigma \not|= K \varepsilon (y, z)$.
Proof

Let $\square \alpha := \square \alpha \land \alpha$, and define

$$\Sigma = \{ y \leq x, x \leq z, x \approx \square x \}$$
Proof

Let $\Box \alpha := \Box \alpha \land \alpha$, and define

$$\Sigma = \{ y \leq x, x \leq z, x \approx \Box x \} \quad \text{and} \quad \Delta = \{ y \leq \Box^k z \mid k \in \mathbb{N} \}.$$
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**Claim.** $\Sigma \models_\mathcal{K} \varepsilon(y, z) \iff \Delta \models_\mathcal{K} \varepsilon(y, z)$.
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Let $\Box \alpha := \Box \alpha \land \alpha$, and define

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George Metcalfe (University of Bern)  Uniform Interpolation  August 2018  14 / 27
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Let $\square \alpha := \Box \alpha \wedge \alpha$, and define

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$$

Claim. $\Sigma \models_{\mathcal{K}} \varepsilon(y, z) \iff \Delta \models_{\mathcal{K}} \varepsilon(y, z)$.

It follows that if $\mathcal{K}$ were coherent, then $\Delta' \models_{\mathcal{K}} \Delta$ for some finite $\Delta' \subseteq \Delta$, and from this that $\mathcal{K} \models \Box^n z \approx \Box^{n+1} z$ for some $n \in \mathbb{N}$, a contradiction.
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Let $\square \alpha := \square \alpha \land \alpha$, and define

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$(\Leftarrow)$ Just observe that $\Sigma \models \mathcal{K} \Delta$.

$(\Rightarrow)$ Suppose that $\Delta \not\models \mathcal{K} \varepsilon(y, z)$. Then there is a complete modal algebra $\mathbf{A}$ and homomorphism $e : \text{Tm}(y, z) \rightarrow \mathbf{A}$ such that $\Delta \subseteq \ker(e)$ and $\varepsilon \not\in \ker(e)$. 

George Metcalfe  (University of Bern) Uniform Interpolation August 2018 14 / 27
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($\Leftarrow$) Just observe that $\Sigma \models_{K} \Delta$.

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Then also $\Sigma \subseteq \ker(e)$, and hence $\Sigma \not\models_{K} \varepsilon(y, z)$.
Can we generalize this proof to other varieties?
Theorem (Kowalski and Metcalfe 2017)

Let $\mathcal{V}$ be a coherent variety of algebras with a meet-semilattice reduct
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Let $\mathcal{V}$ be a coherent variety of algebras with a meet-semilattice reduct and let $t(x, \bar{u})$ be a term satisfying

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Suppose also that for any finitely generated \( A \in \mathcal{V} \) and \( a, \bar{b} \in A \), there exists \( B \in \mathcal{V} \) containing \( A \) as a subalgebra
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\bigwedge_{k \in \mathbb{N}} t^k(a, \bar{b}) = t( \bigwedge_{k \in \mathbb{N}} t^k(a, \bar{b}), \bar{b}).
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Then $\mathcal{V} \models t^n(x, \bar{u}) \approx t^{n+1}(x, \bar{u})$ for some $n \in \mathbb{N}$. 
A normal modal logic $L$ is called **strongly Kripke complete**

E.g., $K$, $KT$, $K4$, $S4$, and $S5$ are strongly Kripke complete, but not $GL$. 

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A normal modal logic $L$ is called **strongly Kripke complete** if for any set of formulas $T \cup \{\alpha\}$,

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E.g., $K$, $KT$, $K4$, $S4$, and $S5$ are strongly Kripke complete, but not $GL$. 
Applying our general criterion with $t(x) = \Box x$, using strong Kripke completeness to establish the fixpoint condition, we obtain:

Theorem
Any coherent strongly Kripke-complete variety of modal algebras is weakly transitive:
that is, it satisfies $\Box^n x \approx \Box^nx$ for some $n \in \mathbb{N}$ (equivalently, it admits equationally definable principal congruences).

Hence a large family of varieties of modal algebras for non-weakly-transitive modal logics, including K and KT, are not coherent, do not admit uniform deductive interpolation, and their theories do not have a model completion.
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We can also show that weakly transitive logics such as K4 and S4 are not coherent.
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\[ t(x, y, z) = \Diamond(y \land \Diamond(z \land x)) \land x. \]

For any normal modal logic \( L \),

\[ \mathcal{V}_L \models t(x, y, z) \leq x \quad \text{and} \quad \mathcal{V}_L \models u \leq v \Rightarrow t(u, y, z) \leq t(v, y, z). \]
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**Lemma**

*Suppose that L admits finite chains:*

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**Lemma**

Suppose that \( L \) admits finite chains: that is, for each \( n \in \mathbb{N} \) there exists a frame \( \langle W, R \rangle \) for \( L \) such that \( |W| = n \) and the reflexive closure of \( R \) is a total order.
Weakly Transitive Modal Logics (1)

We can also show that weakly transitive logics such as K4 and S4 are not coherent using the ternary term

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Lemma

Suppose that L admits finite chains: that is, for each $n \in \mathbb{N}$ there exists a frame $\langle W, R \rangle$ for L such that $|W| = n$ and the reflexive closure of $R$ is a total order. Then $\mathcal{V}_L \not\models t^n(x, y, z) \approx t^{n+1}(x, y, z)$ for all $n \in \mathbb{N}$. 
Theorem

Let $L$ be a normal modal logic admitting finite chains such that $V_L$ is canonical: that is, closed under taking canonical extensions. Then

(a) $V_L$ is not coherent;

(b) $V_L$ does not admit uniform deductive interpolation;

(c) the first-order theory of $V_L$ does not have a model completion.

In particular, $V_K4$ and $V_S4$ are not coherent and do not admit uniform deductive interpolation, and their theories do not have a model completion.
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Theorem

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Ghilardi and Zawadowski (2002) have also proved that no logic extending K4 that has the finite model property and admits all finite reflexive chains and the two-element cluster is coherent.

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However, their methods are more complicated and less general than ours; they also yield similar but incomparable results.
Any locally finite variety (e.g., Boolean algebras, Sugihara monoids, etc.) is coherent
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The variety of groups is not coherent, however, since every finitely generated recursively presented group embeds into some finitely presented group (Higman 1961).
Theorem (Schmidt 1981)

The variety $\mathcal{L}AT$ of lattices is not coherent, does not admit uniform deductive interpolation, and its theory does not have a model completion.
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The variety $\mathcal{LAT}$ of lattices is not coherent, does not admit uniform deductive interpolation, and its theory does not have a model completion.

We obtain an easy proof of this result using our criterion with the term

$$t(x, u_1, u_2, u_3) = (u_1 \land (u_2 \lor (u_3 \land x))) \land x.$$
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- \( \text{LAT} \nvdash t^n(x, \bar{u}) \approx t^{n+1}(x, \bar{u}) \) for each \( n \in \mathbb{N} \).
A **residuated lattice** is an algebraic structure $\langle A, \wedge, \vee, \cdot, \backslash, \slash, e \rangle$ such that $\langle A, \wedge, \vee \rangle$ is a lattice, $\langle A, \cdot, e \rangle$ is a monoid, and for all $a, b, c \in A$,

$$b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c / b.$$
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b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c / b.
\]

Applying our criterion with the term \( t(x) = (x \wedge e)^2 \), we obtain

**Theorem**

*Any coherent variety of residuated lattices that is closed under canonical extensions satisfies* \((x \wedge e)^{n+1} \approx (x \wedge e)^n \) *for some* \( n \in \mathbb{N} \).*
A residuated lattice is an algebraic structure \( \langle A, \land, \lor, \cdot, \setminus, \times, e \rangle \) such that \( \langle A, \land, \lor \rangle \) is a lattice, \( \langle A, \cdot, e \rangle \) is a monoid, and for all \( a, b, c \in A \),

\[
b \leq a \setminus c \iff a \cdot b \leq c \iff a \leq c / b.
\]

Applying our criterion with the term \( t(x) = (x \land e)^2 \), we obtain

**Theorem**

Any coherent variety of residuated lattices that is closed under canonical extensions satisfies \((x \land e)^{n+1} \approx (x \land e)^n\) for some \( n \in \mathbb{N} \).

It follows that varieties of residuated lattices corresponding to the most commonly studied substructural logics are not coherent, and do not admit uniform deductive interpolation.
Our general criterion shows that in a coherent variety with a semilattice reduct, terms satisfying certain conditions admit **fixpoints**.
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Might it be the case that, conversely, admitting such fixpoints **guarantees** the coherence of the variety?
Challenge 1: Understanding Fixpoints

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Might it be the case that, conversely, admitting such fixpoints guarantees the coherence of the variety?

Indeed for certain fixpoint modal logics, the fixpoint operators have been used to construct uniform interpolants.

We have seen that the most well-studied modal and substructural logics, and many important varieties from algebra, are *not* coherent.
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Challenge 2: Dealing with Failure

We have seen that the most well-studied modal and substructural logics, and many important varieties from algebra, are not coherent. In such cases, can we determine instead which terms do admit uniform interpolants?

This problem has been considered for certain description logics, using bisimulations to calculate uniform interpolants when they exist.

C. Lutz and F. Wolter
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C. Lutz and F. Wolter

Can we develop similar methods for constructing uniform interpolants for modal logics, lattices, residuated lattices, etc.?
Can we understand the conservative congruence extension property appearing in Wheeler’s theorem as a property of consequence?

**Theorem (Wheeler 1976)**

*The theory of a variety has a model completion if and only if it is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.*
Challenge 3: Understanding Model Completions

Can we understand the conservative congruence extension property appearing in Wheeler’s theorem as a property of consequence?

**Theorem (Wheeler 1976)**

The theory of a variety has a model completion if and only if it is coherent, admits the amalgamation property, and has the conservative congruence extension property for its finitely presented members.

Can we extend Ghilardi and Zawadowski’s theorem to quasivarieties or (positive) universal classes?

**Theorem (Ghilardi and Zawadowski 2002, van Gool et al. 2017)**

If the a variety admits left and right uniform interpolation and the join-semilattice of compact congruences of any finitely generated free algebra is dually Brouwerian, then its theory has a model completion.