Supernilpotence and Higher Dimensional Congruences

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Overview of Talk

1. Commutator Theory, Nilpotence, and Supernilpotence
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2. Higher Dimensional Congruence Relations
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3. A Stronger Term Condition and Commutator
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4. Supernilpotent Taylor Algebras
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2. Higher Dimensional Congruence Relations
3. A Stronger Term Condition and Commutator
4. Supernilpotent Taylor Algebras
5. Supernilpotence Need Not Imply Nilpotence
Commutator Theory

The classical commutator for a universal algebra $\mathbb{A}$ is a binary operation

$$[\cdot, \cdot] : \text{Con}(\mathbb{A})^2 \to \text{Con}(\mathbb{A})$$

that allows one to define abelianness and generalizations of abelianness such as solvability and nilpotence.
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- For example, an algebra $\mathbb{A}$ is said to be abelian if

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[1, 1] = 0.
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- For example, an algebra $\mathbb{A}$ is said to be **abelian** if
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- The higher commutator is a higher arity operation that generalizes the binary commutator, e.g.
  \[ [\cdot, \ldots, \cdot] : \text{Con}(\mathbb{A})^n \to \text{Con}(\mathbb{A}) \]
Nilpotence and Supernilpotence

Definition

Let $A$ be an algebra and let $\theta \in \text{Con}(A)$. Set

$\theta_0 = \theta$ and

$\theta_{i+1} = [\theta_i, \theta]$ and ($\theta_{i+1} = ([\theta_i, \theta]^TC)$.

These produce two descending chains of congruences, called the derived and lower central series, respectively.

1. If $\theta_n = 0$ then $\theta$ is said to be $n$-step solvable.
2. If ($\theta_n = 0$, then $\theta$ is said to be $n$-step nilpotent.
3. If $\theta$ is such that $\theta, \ldots, \theta \in \text{Con}(A)$, then $\theta$ is said to be $n$-step supernilpotent.
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  1. A finite Mal’cev algebra of finite type is supernilpotent if and only if it is the product of prime power order nilpotent algebras. (Freese & McKenzie, Kearnes, Aichinger & Mudrinski)
  2. There is a polynomial time algorithm to solve the equation satisfiability problem for a finite supernilpotent Mal’cev algebra of finite type. (Kompatscher)
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- Kearnes and Szendrei have announced that any finite supernilpotent algebra is nilpotent.
- It follows from results of Wires that any supernilpotent algebra generating a modular variety is nilpotent.
- We can show any supernilpotent Taylor algebra is nilpotent. (A Taylor algebra is an algebra that satisfies some nontrivial idempotent Mal’cev condition.)
- Moore and M. have constructed a supernilpotent algebra that is not solvable and hence not nilpotent. Note, this algebra is necessarily infinite and not Taylor.
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  2. properties of a relation, usually called $\Delta$. 

Definition (Term Condition)

Let $A$ be an algebra and take $\alpha, \beta, \delta \in \text{Con}(A)$. We say that $\alpha$ centralizes $\beta$ modulo $\delta$ when the following condition is met:

For all $t \in \text{Pol}(A)$ and $a_0 \equiv \alpha b_0$ and $a_1 \equiv \beta b_1$ with $|a_0| + |a_1| = \sigma(t)$, 

$\Rightarrow t(b_0, a_0) \equiv \delta t(b_0, b_1)$

We write $C_{TC}(\alpha, \beta; \delta)$ whenever this is true.

The term condition may be described as a condition that is quantified over a certain invariant relation of $A$ which is called the algebra of $(\alpha, \beta)$-matrices and is denoted $M(\alpha, \beta)$. 
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Matrices

- A square is the graph \( \langle 2^2; E \rangle \), where two functions \( f, g \in 2^2 \) are connected by an edge if and only if their outputs differ in exactly one argument.
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- We say that a relation $R$ on a set $A$ is 2-dimensional if $R \subseteq A^{2^2}$ ($R$ is a set of squares whose vertices are labeled by elements of $A$.)
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- We say that a relation $R$ on a set $A$ is 2-dimensional if $R \subseteq A^{22}$ ($R$ is a set of squares whose vertices are labeled by elements of $A$.)

- $M(\alpha, \beta)$ is the subalgebra of $A^{22}$ with generators

$$\left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : x \equiv_\alpha y \right\} \cup \left\{ \begin{bmatrix} y & y \\ x & x \end{bmatrix} : x \equiv_\beta y \right\}$$
Matrices

For $\delta \in \text{Con}(A)$ we have that $\alpha$ centralizes $\beta$ modulo $\delta$ if the implication

\[
\begin{pmatrix}
c \\
\delta \\
a
\end{pmatrix}
\rightarrow
\begin{pmatrix}
c \\
\delta \\
a
\end{pmatrix}
\]

holds for all $(\alpha, \beta)$-matrices. This condition is abbreviated $C_{TC}(\alpha, \beta; \delta)$. 
Matrices

Similarly, we have that $\beta$ centralizes $\alpha$ modulo $\delta$ if the implication

\[
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\rightarrow
\begin{array}{c}
\text{c} \\
\text{d}
\end{array}
\delta
\]

holds for all $(\alpha, \beta)$-matrices. This condition is abbreviated $C_{\mathcal{T}C}(\beta, \alpha; \delta)$. 
The binary commutator is defined to be

$$[\alpha, \beta]_{T_C} = \bigwedge \{\delta : C(\alpha, \beta; \delta)\}$$
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We say that a relation \( R \) on a set \( A \) is 3-dimensional if \( R \subseteq A^{3^2} \) (\( R \) is a set of cubes whose vertices are labeled by elements of \( A \)).
For congruences $\theta_0, \theta_1, \theta_2 \in \text{Con}(\mathbb{A})$, set $M(\theta_0, \theta_1, \theta_2) \leq \mathbb{A}^{2^3}$ to be the subalgebra generated by the following labeled cubes:

$M(\theta_0, \theta_1, \theta_2)$ is called the algebra of $(\theta_0, \theta_1, \theta_2)$-matrices.
For $\delta \in \text{Con}(\mathbb{A})$, we say that $\theta_0, \theta_1$ **centralize** $\theta_2$ **modulo** $\delta$ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:
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\[
\begin{array}{ccc}
\theta_0 & \theta_1 & \theta_2 \\
\delta & & \\
a & b & c \\
& d & e \\
& & f \\
g & h &
\end{array}
\]
Centrality

For $\delta \in \text{Con}(\mathbb{A})$, we say that $\theta_0, \theta_1$ **centralize** $\theta_2$ **modulo** $\delta$ if the following implication holds for all $(\theta_0, \theta_1, \theta_2)$-matrices:

This condition is abbreviated $C_{TC}(\theta_0, \theta_1, \theta_2; \delta)$. 
Here is a picture of $C_{TC}(\theta_1, \theta_2, \theta_0; \delta)$:
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For congruences $\theta_0, \theta_1, \theta_2$ we set

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Higher centrality and the commutator for arity $\geq 4$ are similarly defined.
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- **Observation:** The term condition definition of centrality involving $n$-many congruences $\theta_0, \ldots, \theta_{n-1}$ is a condition that is quantified over $(\theta_0, \ldots, \theta_{n-1})$-matrices, which are certain $n$-dimensional invariant relations

\[ M(\theta_0, \ldots, \theta_{n-1}) \leq A^{2^n} \]

that have generators of the form

\[ f \in 2^n \text{ such that } f(i) = 0 \]

\[ f \in 2^n \text{ such that } f(i) = 1 \]
Consider the $n$-dimensional hypercube $\mathbb{H}_n = \langle 2^n; E \rangle$. For any coordinate $i \in n$, there are two $(n - 1)$-dimensional hyperfaces that are ‘perpendicular’ to $i$: 
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$(\mathbb{H}_n)_3^0$ and $(\mathbb{H}_n)_3^1$
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2. $(\mathbb{H}_n)_1^i = \langle \{ f \in 2^n : f(i) = 1 \} ; E \rangle$. 
Take \( h \in A^{2^n} \). We consider \( h \) as a vertex labeled \( n \)-dimensional hypercube. For any coordinate \( i \in n \), there are two \((n - 1)\)-dimensional vertex labeled hyperfaces that are perpendicular to \( i \), which we denote
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Fact: Suppose $A$ is a member of a permutable variety, and take $(\theta_0, \ldots, \theta_{n-1}) \in \text{Con}(A)^n$. Then,

$$M(\theta_0, \ldots, \theta_{n-1})_i$$

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is a congruence relation, for all $i \in n$.

This leads to a nice characterization of the commutator for permutable varieties.
Theorem (Binary Commutator)

Let $\mathcal{V}$ be a permutable variety and let $\mathbb{A} \in \mathcal{V}$. For $\alpha, \beta \in \text{Con}(\mathbb{A})$, the following are equivalent:

1. $\langle x, y \rangle \in [\alpha, \beta]\text{TC}$
2. $[xy, x] \in M(\alpha, \beta)$
3. $[a, y, a, x] \in M(\alpha, \beta)$ for some $a \in \mathbb{A}$
4. $[xy, b, b] \in M(\alpha, \beta)$ for some $b \in \mathbb{A}$. 
Theorem (Binary Commutator)

Let $\mathcal{V}$ be a permutative variety and let $A \in \mathcal{V}$. For $\alpha, \beta \in \text{Con}(A)$, the following are equivalent:

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4. $\begin{bmatrix} x & y \\ b & b \end{bmatrix} \in M(\alpha, \beta)$ for some $b \in A$. 
Let \( \mathcal{V} \) be a modular variety and let \( \mathbb{A} \in \mathcal{V} \). For \( \alpha, \beta \in \text{Con}(\mathbb{A}) \), define \( \Delta_{\alpha, \beta} \) to be the transitive closure of \( M(\alpha, \beta) \).

Fact: Both \( (\Delta_{\alpha, \beta})_0 \) and \( (\Delta_{\alpha, \beta})_1 \) are congruence relations.
Let $\mathcal{V}$ be a modular variety and let $A \in \mathcal{V}$. For $\alpha, \beta \in \text{Con}(A)$, define $\Delta_{\alpha,\beta}$ to be the transitive closure of $M(\alpha, \beta)_0$.

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3. $\begin{bmatrix} a & y \\ a & x \end{bmatrix} \in \Delta_{\alpha,\beta}$ for some $a \in A$
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Theorem: Let $\mathcal{V}$ be a permutable variety. Take $\theta_0, \theta_1, \theta_2 \in \text{Con}(A)$ for $A \in \mathcal{V}$. The following are equivalent:

(1) $\langle x, y \rangle \in [\theta_0, \theta_1, \theta_2]$

(2) $\frac{\begin{array}{c} x \\ \hline \end{array}}{\begin{array}{c} x \\ \hline \end{array}} \frac{\begin{array}{c} x \\ \hline \end{array}}{\begin{array}{c} y \\ \hline \end{array}} \in M(\theta_0, \theta_1, \theta_2)$

There exist elements of $A$ such that

(3) $\frac{\begin{array}{c} b \\ \hline \end{array}}{\begin{array}{c} a \\ \hline \end{array}} \frac{\begin{array}{c} c \\ \hline \end{array}}{\begin{array}{c} y \\ \hline \end{array}} \in M(\theta_0, \theta_1, \theta_2)$

(4) $\frac{\begin{array}{c} d \\ \hline \end{array}}{\begin{array}{c} d \\ \hline \end{array}} \frac{\begin{array}{c} x \\ \hline \end{array}}{\begin{array}{c} y \\ \hline \end{array}} \in M(\theta_0, \theta_1, \theta_2)$

(5) $\frac{\begin{array}{c} h \\ \hline \end{array}}{\begin{array}{c} x \\ \hline \end{array}} \frac{\begin{array}{c} i \\ \hline \end{array}}{\begin{array}{c} y \\ \hline \end{array}} \in M(\theta_0, \theta_1, \theta_2)$
Theorem: Let $\mathcal{V}$ be a modular variety. Take $\theta_0, \theta_1, \theta_2 \in \text{Con}(A)$ for $A \in \mathcal{V}$. The following are equivalent:

1. $\langle x, y \rangle \in [\theta_0, \theta_1, \theta_2]$

2. $\begin{array}{c} x \\ x \\ x \end{array} \in \Delta_{\theta_0, \theta_1, \theta_2}$

There exist elements of $A$ such that

3. $\begin{array}{c} b \\ c \\ d \\ e \\ f \end{array} \in \Delta_{\theta_0, \theta_1, \theta_2}$

4. $\begin{array}{c} b \\ c \\ d \\ e \\ f \end{array} \in \Delta_{\theta_0, \theta_1, \theta_2}$

5. $\begin{array}{c} h \\ i \\ j \end{array} \in \Delta_{\theta_0, \theta_1, \theta_2}$
Definition
Let $R \subseteq A^{2^n}$ be an $n$-dimensional relation on some set $A$. $R$ is called an $n$-dimensional equivalence relation if for all $i \in n$, each $R_i$ is an equivalence relation.
Higher Dimensional Congruence Relations

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Let $A$ be an algebra with underlying set $A$. Let $R \in A^{2^n}$ be an $n$-dimensional equivalence relation. $R$ is called an $n$-dimensional congruence if $R$ is preserved by the basic operations of $A$.
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Definition
Let $\mathbb{A}$ be an algebra with underlying set $A$. Let $R \in A^{2^n}$ be an $n$-dimensional equivalence relation. $R$ is called an $n$-dimensional congruence if $R$ is preserved by the basic operations of $\mathbb{A}$.

- Fix $n \geq 1$. The collection of all $n$-dimensional congruences of an algebra $\mathbb{A}$ is an algebraic lattice, which we denote by $\text{Con}_n(\mathbb{A})$.
- There are $n$ distinct embeddings from $\text{Con}_1(\mathbb{A})$ into $\text{Con}_n(\mathbb{A})$. 
\[ \phi_2^0(\alpha) = \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : \langle x, y \rangle \in \alpha \right\} \]
\( \phi_0^2(\alpha) = \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : \langle x, y \rangle \in \alpha \right\} \)

\( \phi_1^2(\beta) = \left\{ \begin{bmatrix} x & x \\ y & y \end{bmatrix} : \langle x, y \rangle \in \beta \right\} \)
Define $\Delta_{\alpha,\beta} = \phi^0_2(\alpha) \lor \phi^1_2(\beta)$

$$\phi^0_2(\alpha) = \left\{ \begin{bmatrix} x & y \\ x & y \end{bmatrix} : \langle x, y \rangle \in \alpha \right\}$$

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Higher Dimensional Congruence Relations

- Fix a dimension $n$ and take $i \in n$. For a pair $\langle x, y \rangle \in A^2$, let $\text{Cube}_i(\langle x, y \rangle) \in A^{2^n}$ be such that
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1. $(\text{Cube}_i(\langle x, y \rangle))_i^0$ is the $(n - 1)$-dimensional cube with each vertex labeled by $x$. 

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Higher Dimensional Congruence Relations
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- Fix a dimension $n$ and take $i \in n$. For a pair $\langle x, y \rangle \in A^2$, let $\text{Cube}_i(\langle x, y \rangle) \in A^{2^n}$ be such that
  1. $(\text{Cube}_i(\langle x, y \rangle))^0_i$ is the $(n - 1)$-dimensional cube with each vertex labeled by $x$.
  2. $(\text{Cube}_i(\langle x, y \rangle))^1_i$ is the $(n - 1)$-dimensional cube with each vertex labeled by $y$.

- Define $\phi^i_n : \text{Con}_1(\mathbb{A}) \to \text{Con}_n(\mathbb{A})$ by
  $$\phi^i_n(\alpha) = \{\text{Cube}_i(\langle x, y \rangle) : \langle x, y \rangle \in \alpha\}$$
Define $\Delta_{\theta_0, \ldots, \theta_{n-1}} = \bigvee_i \phi_n^i(\theta_i)$
Characterizing Joins

Let $\mathbb{A}$ be an algebra and let $\theta$ be an equivalence relation on $\mathbb{A}$. Then, $\theta$ is an admissible relation if and only if $\theta$ is compatible with the unary polynomials of $\mathbb{A}$. This generalizes to:

**Theorem**

Let $\mathbb{A}$ be an algebra and let $n \geq 1$. An $n$-dimensional equivalence relation $\theta$ is admissible if and only if $\theta$ is compatible with the $n$-ary polynomials of $\mathbb{A}$.
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**Theorem**

Let $\mathcal{A}$ be an algebra and let $n \geq 1$. An $n$-dimensional equivalence relation $\theta$ is admissible if and only if $\theta$ is compatible with the $n$-ary polynomials of $\mathcal{A}$.
Proof Idea

Take $a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3, c_0, d_0, c_1, d_1, c_2, d_2, c_3, d_3 \in \theta$
Proof Idea

Take \( a_0 \rightarrow b_0 \rightarrow a_1 \rightarrow b_1 \rightarrow a_2 \rightarrow b_2 \rightarrow a_3 \rightarrow b_3 \quad \in \theta \quad \)

Then,

\( c_0 \rightarrow d_0 \rightarrow c_1 \rightarrow d_1 \rightarrow c_2 \rightarrow d_2 \rightarrow c_3 \rightarrow d_3 \rightarrow \in \theta \)
Proof Idea

Take \( a_0 \longrightarrow b_0 \longrightarrow a_1 \longrightarrow b_1 \longrightarrow a_2 \longrightarrow b_2 \longrightarrow a_3 \longrightarrow b_3 \), \( c_0 \longrightarrow d_0 \), \( c_1 \longrightarrow d_1 \), \( c_2 \longrightarrow d_2 \), \( c_3 \longrightarrow d_3 \) \( \in \theta \) Then, \( a_1 \longrightarrow a_1 \longrightarrow b_1 \longrightarrow b_1 \longrightarrow b_1 \), \( a_1 \longrightarrow a_1 \longrightarrow b_1 \longrightarrow b_1 \longrightarrow b_1 \), \( c_1 \longrightarrow c_1 \longrightarrow d_1 \longrightarrow d_1 \longrightarrow d_1 \), \( c_1 \longrightarrow c_1 \longrightarrow d_1 \longrightarrow d_1 \longrightarrow d_1 \), \( \in \theta \) Compatibility with binary polynomials is sufficient to show compatibility with a 4-ary operation.
Characterizing Joins

\[ \Delta_{\theta_0, \ldots, \theta_{n-1}} = \bigvee_i \phi^i_n(\theta_i) \]
is therefore obtained by

1. Closing \( \bigcup \phi^i_n(\theta_i) \) under all \( n \)-ary polynomials and then

\[ M(\theta_0, \ldots, \theta_{n-1}) \leq \Delta_{\theta_0, \ldots, \theta_{n-1}} \]. We use this larger collection to define a stronger term condition.
Characterizing Joins

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is therefore obtained by

1. Closing \( \bigcup \phi^i_n(\theta_i) \) under all \( n \)-ary polynomials and then
2. taking a sequence of transitive closures, cycling through all
   possible directions possibly \( \omega \)-many times.

Notice: \( M(\theta_0, \ldots, \theta_{n-1}) \leq \Delta_{\theta_0, \ldots, \theta_{n-1}} \). We use this larger
   collection to define a stronger term condition.
Hypercentrality

For $\delta \in \text{Con}(\mathbb{A})$ we have that $\alpha$ hypercentralizes $\beta$ modulo $\delta$ if the implication holds for all members of $\Delta_{\alpha,\beta}$. This condition is abbreviated $C_H(\alpha, \beta; \delta)$. 
Similarly, we have that $\beta$ hypercentralizes $\alpha$ modulo $\delta$ if the implication

$$\alpha \rightarrow \beta$$

holds for all members of $\Delta_{\alpha,\beta}$. This condition is abbreviated $C_H(\beta, \alpha; \delta)$. 
Hypercentrality

- For congruences $\theta_0, \theta_1$ we set

$$[\theta_0, \theta_1]_H = \bigwedge \{\delta : C_H(\theta_0, \theta_1; \delta)\}$$
Hypercentrality

- For congruences $\theta_0, \theta_1$ we set

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- Higher arity hypercentrality and the higher arity hypercommutator similarly defined.
Theorem (Binary Hyper Commutator)

Let $\mathbb{A}$ be an algebra. For $\alpha, \beta \in \text{Con}(\mathbb{A})$, the following are equivalent:

1. $\langle x, y \rangle \in [\alpha, \beta]_H$

2. $\begin{bmatrix} x & y \\ x & x \end{bmatrix} \in \Delta_{\alpha, \beta}$

3. $\begin{bmatrix} a & y \\ a & x \end{bmatrix} \in \Delta_{\alpha, \beta}$ for some $a \in A$

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A similar characterization of the higher arity hyper commutator also holds.
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Supernilpotent Taylor Algebras Are Nilpotent

Strategy:
1. From the definitions, it follows that
\[
\theta_0, \ldots, \theta_{n-1} \leq \theta_0, \ldots, \theta_{n-1}
\]
2. Demonstrate the commutator nesting property for the hyper commutator:
\[
[\theta_0, \ldots, \theta_{i-1}, \theta_i, \ldots, \theta_{n-1}]_H \leq \theta_0, \ldots, \theta_{n-1}
\]
3. Show that \([\theta, \ldots, \theta]_S = [\theta, \ldots, \theta]_H\) in a Taylor variety.
4. (2) and (3) imply that
\[
[\theta, \ldots, \theta]_T, \theta, \ldots, \theta \leq [\theta, \ldots, \theta]_H \leq [\theta, \ldots, \theta]_T
\]
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2. Demonstrate the commutator nesting property for the hyper commutator:
   \[
   \left[\theta_0, \ldots, \theta_{i-1}\right] \text{H}, \theta_i, \ldots, \theta_{n-1} \text{H} \leq \theta_0, \ldots, \theta_{n-1} \text{H}
   \]

3. Show that $\theta, \ldots, \theta \text{S} = \theta, \ldots, \theta \text{H}$ in a Taylor variety.

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\[[[\theta, \ldots, \theta]_{TC}, \theta, \ldots, \theta]_{TC} = [[[\theta, \ldots, \theta]_H, \theta, \ldots, \theta]_H \leq [\theta, \ldots, \theta]_H = [\theta, \ldots, \theta]_{TC}\]
Supernilpotent $\nRightarrow$ Nilpotent (work with Moore)
Supernilpotent $\iff$ Nilpotent (work with Moore)

Define $A = O \cup R \cup G$ with $G$ infinite, $O = \{o^i_j : i, j \in \omega\}$, and $R = \{r^i_j : i, j \in \omega\}$.
Supernilpotent $\not\iff$ Nilpotent (work with Moore)

Define $A = O \cup R \cup G$ with $G$ infinite, $O = \{o^i_j : i, j \in \omega\}$, and $R = \{r^i_j : i, j \in \omega\}$. Let $A = \langle A; t \rangle$ be the algebra with underlying set $A$ and a binary operation $t$ with the table
Supernilpotent $\iff$ Nilpotent (work with Moore)

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where $t$ an injection into $G$ otherwise.
Supernilpotent $\not\Rightarrow$ Nilpotent

- $\mathbb{A}$ is not solvable and hence not nilpotent.

Question: Let $\mathbb{V}$ be a chapter in the lattice of interpretability of types that does not lie above Olšák's variety. Is there a variety $\mathbb{W} \in \mathbb{V}$ with a supernilpotent algebra that is not nilpotent?
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- $\mathbb{A}$ is 2-step supernilpotent. To prove this it suffices to show that

$$h = \begin{align*}
  &a & b \\
  &c & e \\
  &a & b \\
  &c & d
\end{align*} \in M(1,1,1)$$

implies $e = d$. 

This example generalizes to 'higher dimensions.' There exist algebras $\mathbb{A}_n$ that

1. are not solvable in dimension $n$ (no term in commutators up to arity $n$ evaluated at 1 produces 0)
2. but are $n$-step supernilpotent.

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\[
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 h &= \begin{array}{c}
 a \\
 c \\
 a
\end{array} 
\begin{array}{c}
 b \\
 \cdots \\
 b \\
 a
\end{array} 
\begin{array}{c}
 c \\
 e
\end{array} 
\begin{array}{c}
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\end{array}
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Thank you for attending this presentation.
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