A duality-theoretic approach to MTL-algebras

Sara Ugolini

(Joint work with W. Fussner)

BLAST 2018 - Denver, August 6th 2018
A commutative, integral residuated lattice, or CIRL, is a structure $A = (A, \cdot, \rightarrow, \land, \lor, 1)$ where:

(I) $(A, \land, \lor, 1)$ is a lattice with top element 1,

(II) $(A, \cdot, 1)$ is a commutative monoid,

(III) $(\cdot, \rightarrow)$ is a residuated pair, i.e. it holds for every $x, y, z \in A$:

$$x \cdot z \leq y \iff z \leq x \rightarrow y.$$ 

CIRLs constitute a variety, $\mathbb{RL}$.

Examples: $(\mathbb{Z}^-, +, \ominus, \text{min}, \text{max}, 0)$, ideals of a commutative ring...
A bounded CIRL, or BCIRL, is a CIRL $A = (A, \cdot, \rightarrow, \wedge, \vee, 0, 1)$ with an extra constant 0 that is the least element of the lattice.

Examples: Boolean algebras, Heyting algebras...

In every BRL we can define further operations and abbreviations:

$$\neg x = x \rightarrow 0, \quad x + y = \neg (\neg x \cdot \neg y), \quad x^2 = x \cdot x.$$  

Totally ordered structures are called *chains.*
A bounded CIRL, or BCIRL, is a CIRL $A = (A, \cdot, \rightarrow, \wedge, \vee, 0, 1)$ with an extra constant $0$ that is the least element of the lattice.

Examples: Boolean algebras, Heyting algebras...

In every BRL we can define further operations and abbreviations:

$\neg x = x \rightarrow 0$, $x + y = \neg(\neg x \cdot \neg y)$, $x^2 = x \cdot x$.

 Totally ordered structures are called chains.

A CIRL, or BCIRL, is semilinear (or prelinear, or representable) if it is a subdirect product of chains.

We call semilinear CIRLs GMTL-algebras and semilinear BCIRLs MTL-algebras. They constitute varieties that we denote with GMTL and MTL.

MTL-algebras are the semantics of Esteva and Godo’s MTL, the fuzzy logic of left-continuous t-norms.
**Priestley Duality**

MTL-algebras and GMTL-algebras have a **distributive** lattice reduct.

[Priestley, 1970]: The category \( \text{BDL} \) of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category \( \text{Pries} \) of Priestley spaces and continuous isotone maps.

A **Priestley space** is a structure \((X, \leq, \tau)\), where \((X, \tau)\) is a compact topological space, \((X, \leq)\) is a poset, and for any \(x \not\leq y\) there exists a clopen \(U \subseteq X\) with \(x \in U\) and \(y \notin U\).
**Priestley duality**

MTL-algebras and GMTL-algebras have a **distributive** lattice reduct.

[Priestley, 1970]: The category $\mathbf{BDL}$ of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category $\mathbf{Pries}$ of Priestley spaces and continuous isotone maps.

A **Priestley space** is a structure $(X, \leq, \tau)$, where $(X, \tau)$ is a compact topological space, $(X, \leq)$ is a poset, and for any $x \not\leq y$ there exists a clopen $U \subseteq X$ with $x \in U$ and $y \notin U$.

![Diagram](attachment:diagram.png)
Priestley duality

MTL-algebras and GMTL-algebras have a distributive lattice reduct.

[Priestley, 1970]: The category $\mathbb{BDL}$ of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category $\mathbb{Pries}$ of Priestley spaces and continuous isotone maps.

A Priestley space is a structure $(X, \leq, \tau)$, where $(X, \tau)$ is a compact topological space, $(X, \leq)$ is a poset, and for any $x \nleq y$ there exists a clopen $U \subseteq X$ with $x \in U$ and $y \notin U$.

\[ S(D): \text{prime filters ordered by inclusion with topology generated by} \]
\[ \{\varphi(d) : d \in D\} \cup \{\varphi(d)^c : d \in D\} \]
where $\varphi(d) = \{X \text{ prime filter of } D : d \in X\}$
**Priestley duality**

MTL-algebras and GMTL-algebras have a *distributive* lattice reduct.

[Priestley, 1970]: The category $\mathbf{BDL}$ of bounded distributive lattices and bounded lattice homomorphisms is dually equivalent to the category $\mathbf{Pries}$ of Priestley spaces and continuous isotone maps.

A *Priestley space* is a structure $(X, \leq, \tau)$, where $(X, \tau)$ is a compact topological space, $(X, \leq)$ is a poset, and for any $x \nleq y$ there exists a clopen $U \subseteq X$ with $x \in U$ and $y \notin U$.

\[
\begin{array}{ccc}
\mathbf{BDL} & \xrightarrow{\mathcal{A}} & \mathbf{Pries} \\
& \mathcal{S} \swarrow & \searrow \mathcal{S} \\
\end{array}
\]

$\mathcal{S}(D)$: prime filters ordered by inclusion with topology generated by

\[
\{\varphi(d) : d \in D\} \cup \{\varphi(d)^c : d \in D\}
\]

where $\varphi(d) = \{X \text{ prime filter of } D : d \in X\}$

$\mathcal{A}(X, \leq, \tau)$: collection of clopen upsets $(\text{Cl}(X), \cup, \cap, \emptyset, X)$
Priestley duality admits numerous modifications. E.g. if one or both of the lattice bounds are dropped we obtain a dual category of pointed, or doubly-pointed, (i.e. bounded above or bounded) Priestley spaces.
Priestley duality admits numerous modifications. E.g. if one or both of the lattice bounds are dropped we obtain a dual category of pointed, or doubly-pointed, (i.e. bounded above or bounded) Priestley spaces.

Moreover, Priestley duality can be extended to distributive residuated lattices.

Our approach is essentially drawn from [Galatos, PhD thesis, 2003] and [Urquhart, 1996], however a similar approach to duals of MTL-algebras has been developed by Cabrer and Celani in 2006.

Usually, the multiplication is dualized as a ternary relation on prime filters.
**Residuated spaces**

We call a structure \((S, R, E)\) an unpointed residuated space if

- \(S\) is a Priestley space
- \(R\) is a ternary relation on \(S\),
- \(E\) is a subset of \(S\),
- for all \(x, y, z, w, x', y', z' \in S\) and \(U, V \in A(S)\):
  
  (I) \(R(x, y, u)\) and \(R(u, z, w)\) for some \(u \in S\) if and only if \(R(y, z, v)\) and \(R(x, v, w)\) for some \(v \in S\).
  
  (II) If \(x' \leq x\), \(y' \leq y\), and \(z \leq z'\), then \(R(x, y, z)\) implies \(R(x', y', z')\).
  
  (III) If \(R(x, y, z)\), then there exist \(U, V \in A(S)\) such that \(x \in U\), \(y \in V\), and \(z \notin R[U, V, -]\).
  
  (IV) For all \(U, V \in A(S)\), each of \(R[U, V, -]\), \(\{z \in S : R[z, V, -] \subseteq U\}\), and \(\{z \in P : R[B, z, -] \subseteq U\}\) are clopen.
  
  (V) \(E \in A(S)\) and for all \(U \in A(S)\) we have \(R[U, E, -] = R[E, U, -] = U\).

Where \(R[U, V, -] = \{z \in S : (\exists x \in U)(\exists y \in V)(R(x, y, z))\}\).
Residuated spaces

If $S_1 = (S_1, \leq_1, \tau_1, R_1, E_1)$ and $S_2 = (S_2, \leq_2, \tau_2, R_2, E_2)$ are unpointed residuated spaces, a map $\alpha : S_1 \to S_2$ is a bounded morphism if:

(I) $\alpha$ is a continuous isotone map.

(II) If $R_1(x, y, z)$, then $R_2(\alpha(x), \alpha(y), \alpha(z))$.

(III) If $R_2(u, v, \alpha(z))$, then there exist $x, y \in S_1$ such that $u \leq \alpha(x)$, $v \leq \alpha(y)$, and $R_1(x, y, z)$.

(IV) For all $U, V \in A(S_2)$ and all $x \in S_1$, if $R_1[x, \alpha^{-1}[U], \neg] \subseteq \alpha^{-1}[V]$, then $R_2[\alpha(x), U, \neg] \subseteq V$.

(V) $\alpha^{-1}[E_2] \subseteq E_1$.

We denote the category of unpointed residuated spaces and bounded morphisms by $uRS$. The following is proven in [Galatos, PhD thesis].

Theorem

The category of bounded residuated lattices with residuated lattice homomorphisms preserving the lattice bounds is dually equivalent to $uRS$. 

EXTENDING FUNCTORS

• Given an unpointed residuated space $S = (S, \leq, \tau, R, E)$, we define $A(S) = (A(S, \leq, \tau), \cdot, \rightarrow, E)$, where

$$U \cdot V = R[U, V, -]$$

$$U \rightarrow V = \{ x \in S : R[x, U, -] \subseteq V \}$$

for $U, V \in A(S, \leq, \tau)$, where

$$R[U, V, -] = \{ z \in S : (\exists x \in U)(\exists y \in V)(R(x, y, z)) \}.$$
EXTENDING FUNCTORS

- Given an unpointed residuated space $S = (S, \leq, \tau, R, E)$, we define $A(S) = (A(S, \leq, \tau), \cdot, \rightarrow, E)$, where

  $$U \cdot V = R[U, V, -]$$
  $$U \rightarrow V = \{x \in S : R[x, U, -] \subseteq V\}$$

  for $U, V \in A(S, \leq, \tau)$, where

  $$R[U, V, -] = \{z \in S : (\exists x \in U)(\exists y \in V)(R(x, y, z))\}.$$

- Given a BCIRL $A$, we define a product $\bullet$ on prime filters as the upset of the complex product $\cdot$:

  $$a \bullet b = \uparrow(a \cdot b) = \{z \in A : \exists x \in a, y \in b, xy \leq z\}$$

  $S(A) = (S(D), R, E)$, where for a bounded residuated lattice $A$ with bounded lattice reduct $D$, we define a ternary relation $R$ on $S(D)$ and a subset of $S(D)$ by

  $$R(a, b, c) \text{ iff } a \bullet b \subseteq c,$$

  $$E = \{a \in S(D) : 1 \in a\}.$$
Duality for MTL

Let $A = (A, \wedge, \vee, \cdot, \backslash, /, 1, \bot, \top)$ be a bounded residuated lattice and $S = (S, \leq, \tau, R, E)$ its dual space.

- $A$ is commutative iff for all $x, y, z \in S$, $R(x, y, z)$ iff $R(y, x, z)$.

- $A$ is integral iff $E = S$.

- In the presence of integrality and commutativity, $A$ is semilinear iff for all $x, y, z, v, w \in S$, if $R(x, y, z)$ and $R(x, v, w)$, then $y \leq w$ or $v \leq z$.

We denote by $\mathbf{MTL}^\tau$ the full subcategory of $\mathbf{uRS}$ whose objects satisfy these three conditions.

**Theorem**

$\mathbf{MTL}^\tau$ is dually equivalent to $\mathbf{MTL}$. 
Duality for GMTL

Let $\text{MTL}_{\text{div}}$ be the full subcategory of $\text{MTL}$-algebras without zero divisors and $\text{SMTL}_{\text{ind}}$ be the full subcategory of directly indecomposable $\text{SMTL}$-algebras.

**Theorem**

The categories GMTL, $\text{MTL}_{\text{div}}$, and $\text{SMTL}_{\text{ind}}$ are equivalent.
Duality for GMTL

Let $\text{MTL}_{\text{div}}$ be the full subcategory of $\text{MTL}$-algebras without zero divisors and $\text{SMTL}_{\text{ind}}$ be the full subcategory of directly indecomposable $\text{SMTL}$-algebras.

**THEOREM**

The categories $\text{GMTL}$, $\text{MTL}_{\text{div}}$, and $\text{SMTL}_{\text{ind}}$ are equivalent.

We provide a duality for $\text{GMTL}$, using as a bridge the full subcategory $\text{MTL}_{\text{div}}^\tau$, that is characterized by having a greatest element $\top$. 
Duality for GMTL

Let \( \mathbf{MTL}_{\text{div}} \) be the full subcategory of \( \mathbf{MTL} \)-algebras without zero divisors and \( \mathbf{SMTL}_{\text{ind}} \) be the full subcategory of directly indecomposable \( \mathbf{SMTL} \)-algebras.

**Theorem**

The categories \( \mathbf{GMTL}, \mathbf{MTL}_{\text{div}}, \) and \( \mathbf{SMTL}_{\text{ind}} \) are equivalent.

We provide a duality for \( \mathbf{GMTL} \), using as a bridge the full subcategory \( \mathbf{MTL}_{\text{div}}^\tau \), that is characterized by having a greatest element \( \top \).

Thus, let \( \mathbf{GMTL}^\tau \) be the category whose objects are of the form \( (S, R, E, \top) \), where \( (S, R, E) \) is an object of \( \mathbf{MTL}^\tau \) with a \( \top \). The morphisms are bounded morphisms that preserve \( \top \).

**Theorem**

\( \mathbf{GMTL} \) and \( \mathbf{GMTL}^\tau \) are dually equivalent.
Dualities for classes of distributive residuated lattices typically employ a ternary relation on dual structures.

For prelinear residuated structures, the ternary relation on duals may be presented by means of a (possibly partially-defined) binary operation $\bullet$ on the underlying Priestley dual of the lattice reduct.
Dualities for classes of distributive residuated lattices typically employ a ternary relation on dual structures.

For prelinear residuated structures, the ternary relation on duals may be presented by means of a (possibly partially-defined) binary operation \( \bullet \) on the underlying Priestley dual of the lattice reduct.

In a general setting, the *functionality* of duals is treated in [Gehrke, 2016] and [Fussner, Palmigiano 2018]. We explore the cases of MTL and GMTL.
Dualities for classes of distributive residuated lattices typically employ a ternary relation on dual structures.

For prelinear residuated structures, the ternary relation on duals may be presented by means of a (possibly partially-defined) binary operation \( \bullet \) on the underlying Priestley dual of the lattice reduct.

In a general setting, the *functionality* of duals is treated in [Gehrke, 2016] and [Fussner, Palmigiano 2018]. We explore the cases of MTL and GMTL.

**Lemma**

*Let \( \mathbf{A} \) be a GMTL-algebra or an MTL-algebra, and let \( \alpha, \beta \) be nonempty filters of \( \mathbf{A} \). Then if either one of \( \alpha \) or \( \beta \) is prime, we have that either \( \alpha \bullet \beta \) is prime or \( \alpha \bullet \beta = \mathbf{A} \).*
Dualities for classes of distributive residuated lattices typically employ a ternary relation on dual structures.

For prelinear residuated structures, the ternary relation on duals may be presented by means of a (possibly partially-defined) binary operation \( \bullet \) on the underlying Priestley dual of the lattice reduct.

In a general setting, the *functionality* of duals is treated in [Gehrke, 2016] and [Fussner, Palmigiano 2018]. We explore the cases of MTL and GMTL.

**Lemma**

*Let \( A \) be a GMTL-algebra or an MTL-algebra, and let \( a, b \) be nonempty filters of \( A \). Then if either one of \( a \) or \( b \) is prime, we have that either \( a \bullet b \) is prime or \( a \bullet b = A \).*

**Corollary**

*If \( A \) is an GMTL-algebra, then \( \bullet \) is a binary operation on \( S(A) \). If \( A \) is an MTL-algebra, then \( \bullet \) is gives a partial operation on \( S(A) \) and is undefined for \( a, b \in S(A) \) only if \( a \bullet b = A \).*
The following provides a mechanism for defining $\bullet$ on $\text{MTL}^\tau$ and $\text{GMTL}^\tau$.

**Lemma**

Let $S = (S, \leq, \tau, R, E)$ be an object of $\text{MTL}^\tau$. If $x, y, z \in S$ with $R(x, y, z)$, then there exists a least element $z' \in S$ such that $R(x, y, z')$. If $S$ is in $\text{GMTL}^\tau$, then for any $x, y \in S$ there exists a least $z' \in S$ with $R(x, y, z')$.

\[
x \bullet y = \begin{cases} 
\min\{z \in S : R(x, y, z)\}, & \text{if } \{z \in S : R(x, y, z)\} \neq \emptyset \\
\text{undefined}, & \text{otherwise}
\end{cases}
\]
The following provides a mechanism for defining $\bullet$ on $\text{MTL}^\tau$ and $\text{GMTL}^\tau$.

**Lemma**

Let $S = (S, \leq, \tau, R, E)$ be an object of $\text{MTL}^\tau$. If $x, y, z \in S$ with $R(x, y, z)$, then there exists a least element $z' \in S$ such that $R(x, y, z')$. If $S$ is in $\text{GMTL}^\tau$, then for any $x, y \in S$ there exists a least $z' \in S$ with $R(x, y, z')$.

\[
x \bullet y = \begin{cases} 
\min\{z \in S : R(x, y, z)\}, & \text{if } \{z \in S : R(x, y, z)\} \neq \emptyset \\
\text{undefined}, & \text{otherwise}
\end{cases}
\]

**Lemma**

Let $S$ be an object of $\text{MTL}^\tau$ or $\text{GMTL}^\tau$, and let $x, y, z \in S$.

(i) $R(x, y, z)$ iff $x \bullet y \leq z$.

(ii) Each of the following holds (whenever $\bullet$ is defined).

1. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$.
2. $x \bullet y = y \bullet x$.
3. $x \leq y$ implies that $x \bullet z \leq y \bullet z$ and $z \bullet x \leq z \bullet y$. 
Moreover, \( \cdot \) possesses a partial residual.

**Proposition**

Let \( \mathbf{A} \) be an MTL-algebra or GMTL-algebra, and suppose that \( b, c \in S(\mathbf{A}) \) are such that there exists \( a \in S(\mathbf{A}) \) with \( a \cdot b \subseteq c \). Then there is a greatest such \( a \), and it is given by

\[
    b \Rightarrow c := \bigcup \{ a \in S(\mathbf{A}) : a \cdot b \subseteq c \}.
\]

Moreover, \( a \cdot b \subseteq c \) if and only if \( a \subseteq b \Rightarrow c \).
Moreover, \( \bullet \) possesses a partial residual.

**Proposition**

Let \( A \) be an MTL-algebra or GMTL-algebra, and suppose that \( b, c \in S(A) \) are such that there exists \( a \in S(A) \) with \( a \bullet b \subseteq c \). Then there is a greatest such \( a \), and it is given by

\[
b \Rightarrow c := \bigcup \{ a \in S(A) : a \bullet b \subseteq c \}.
\]

Moreover, \( a \bullet b \subseteq c \) if and only if \( a \subseteq b \Rightarrow c \).

**Corollary**

Let \( S \) be an object of MTL\(^\tau\) or GMTL\(^\tau\), and suppose that \( y, z \in S \) are such that there exists \( x \in S \) with \( R(x, y, z) \). Then there is a greatest such \( x \), which we denote by \( y \Rightarrow z \). Moreover, \( x \bullet y \leq z \) if and only if \( x \leq y \Rightarrow z \).
In order to treat negation, we use the Routley star ([Routley, Meyer - 1972,1973], [Urquhart, 1996]). If $A$ is an MTL-algebra and $a \in S(A)$, we define

$$a^* = \{a \in A : \neg a \notin a\}$$

It is easy to see that if $a$ is a prime filter, then so is $a^*$. Moreover, $^*$ is an order-reversing operation on prime filters.
In order to treat negation, we use the Routley star ([Routley, Meyer - 1972,1973], [Urquhart, 1996]). If $A$ is an $\mathbb{MTL}$-algebra and $a \in S(A)$, we define

$$a^* = \{a \in A : \neg a \not\in a\}$$

It is easy to see that if $a$ is a prime filter, then so is $a^*$. Moreover, $*$ is an order-reversing operation on prime filters.

**Lemma**

Let $A$ be an $\mathbb{MTL}$-algebra and let $a \in S(A)$. Then $a^*$ is the largest prime filter of $A$ such that $a \cdot a^* \neq A$.

**Corollary**

Let $S$ be an object of $\mathbb{MTL}^\tau$, and let $x \in S$. Then there exists a greatest $y \in S$ such that $R(x, y, z)$ for some $z \in S$. Equivalently, there exists a greatest $y \in S$ such that $x \cdot y$ is defined.

In light of the previous corollary, for an abstract object $S$ of $\mathbb{MTL}^\tau$ we define for any $x \in S$,

$$x^* := \max\{y \in S : \exists z \in S, R(x, y, z)\}.$$
We now focus on a special class of MTL-algebras: \textit{srDL}-algebras, that are the axiomatic extension of MTL by:

\[(x + x)^2 = x^2 + x^2 \quad (DL)\]
\[\neg(x^2) \to (\neg\neg x \to x) = 1 \quad (r)\]

Relevant subvarieties are: product algebras, Gödel algebras, the variety generated by perfect MV-algebras, nilpotent minimum without negation fixpoint, pseudocomplemented MTL algebras...

Let $\mathbf{R} = (R, \cdot, \to, \land, \lor, 1)$ be a GMTL-algebra, and let $\delta : R \to R$ be a wdl-admissible operator:

- closure operator;
- $\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)$ (nucleus on $\mathbf{R}$);
- $\delta(x \lor y) = \delta(x) \lor \delta(y)$,
  $\delta(x \land y) = \delta(x) \land \delta(y)$.
Let $\mathcal{R} = (\mathcal{R}, \cdot, \rightarrow, \wedge, \vee, 1)$ be a GMTL-algebra, and let $\delta : \mathcal{R} \rightarrow \mathcal{R}$ be a wdl-admissible operator:

- closure operator;
- $\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)$ (nucleus on $\mathcal{R}$);
- $\delta(x \vee y) = \delta(x) \vee \delta(y)$,
  $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$.

Examples: $\delta_D = id$, $\delta_L = \bar{1}$, i.e. $\delta_L(x) = 1$ for every $x \in \mathcal{R}$. 
Let $R = (R, \cdot, \to, \land, \lor, 1)$ be a GMTL-algebra, and let $\delta : R \to R$ be a wdl-admissible operator:

- closure operator;
- $\delta(x) \cdot \delta(y) \leq \delta(x \cdot y)$ (nucleus on $R$);
- $\delta(x \lor y) = \delta(x) \lor \delta(y)$,
  $\delta(x \land y) = \delta(x) \land \delta(y)$.

Examples: $\delta_D = id$, $\delta_L = \overline{1}$, i.e. $\delta_L(x) = 1$ for every $x \in R$.

We define the $\delta$-rotation $R^{\delta}(R)$ as the structure with domain

$$(\{1\} \times R) \cup (\{0\} \times \delta[R])$$

and suitably defined operations.
NOTABLE SUBVARIETIES

With $\delta = id$ we get directly indecomposable $s\mathbb{IDL}$-algebras (introduced as $\mathbb{IBP}_0$-algebras in [Noguera, Esteva, Gispert - 2005]), i.e. disconnected rotation (as defined by Jenei) of GMTL-algebras.

Examples: the variety generated by perfect MV-algebras, $\mathbb{NM}^-$ of nilpotent minimum without negation fixpoint.
Notable Subvarieties

With $\delta = \bar{1}$ we get directly indecomposable pseudocomplemented MTL-algebras: algebras $A$ isomorphic to $2 \oplus R$ for some GMTL $R$.

Examples: Gödel algebras (prelinear Heyting algebras), Product algebras.
srDL-algebras

[Aguzzoli, Flaminio, U. - 2017]: srDL-algebras are equivalent to categories whose objects are quadruples \((B, R, \vee_e, \delta)\):

- \(B\) is a Boolean algebra,
- \(R\) is a GMTL-algebra,
- \(\delta : R \to R\) is wdl-admissible,
- \(\vee_e : B \times R \to R\) is an external join:

\((V1)\) For fixed \(b \in B\) and \(c \in R\):
\[
\nu_b(x) = b \vee_e x \text{ is an endomorphism of } R,
\gamma_c(x) = x \vee_e c \text{ is a lattice homomorphism from } B \text{ to } R.
\]

\((V2)\) \(\nu_0\) is the identity on \(R\),
\(\nu_1\) is constantly equal to 1.

\((V3)\) For all \(b, b' \in B\) and for all \(c, c' \in R\),
\[
\nu_b(c) \vee \nu_{b'}(c') = \nu_{b \vee b'}(c \vee c') = \nu_b(h_{b'}(c \vee c')).
\]
In analogy to the algebraic side, we expect that if $A$ is an srDL-algebra, $S(A)$ can be reconstructed by:

- the Stone space associated to its Boolean skeleton $\mathcal{B}(A)$;
- the object in $\text{GMTL}^\tau$ associated to $\text{Rad}(A)$;
- the maps induced on the dual side by the (external) join;
- the dual of the wdl-admissible operator $\delta$ (¬¬ of $A$).

Indeed, we can reconstruct $S(A)$ by $F_A$: sets of pairs $(u, y)$ such that:

1. $u$ is an ultrafilter of $\mathcal{B}(A)$,
2. $y$ is a prime lattice filter of $\text{Rad}(A)$,
3. for $b \in \mathcal{B}(A), c \in \text{Rad}(A)$, if $b \lor c \in y$, then $b \in u$ or $c \in y$.
**Dualized Construction**

In analogy to the algebraic side, we expect that if $\mathbf{A}$ is an srDL-algebra, $S(\mathbf{A})$ can be reconstructed by:

- the Stone space associated to its Boolean skeleton $\mathcal{B}(\mathbf{A})$;
- the object in $\text{GMTL}^\tau$ associated to $\text{Rad}(\mathbf{A})$;
- the maps induced on the dual side by the (external) join;
- the dual of the wdl-admissible operator $\delta$ ($\lnot \lnot$ of $\mathbf{A}$).

Indeed, we can reconstruct $S(\mathbf{A})$ by $\mathcal{F}_\mathbf{A}$: sets of pairs $(u, \eta)$ such that

1. $u$ is an ultrafilter of $\mathcal{B}(\mathbf{A})$,
2. $\eta$ is a prime lattice filter of $\text{Rad}(\mathbf{A})$,
3. for $b \in \mathcal{B}(\mathbf{A})$, $c \in \text{Rad}(\mathbf{A})$, if $b \lor c \in \eta$, then $b \in u$ or $c \in \eta$.

plus the information given by $\lnot \lnot$. 
Dualized construction

Let $u, v, w \in S(\mathcal{B}(A))$. 

\[ \bullet \ u \quad \bullet \ v \quad \bullet \ w \]
**Dualized construction**

Below $u$: $x \in S(\mathcal{R}(A))$ that respect the “external primality condition” wrt $u$, ordered by inclusion.
**Dualized construction**

Same for $v, w, \ldots$
Dualized construction

Rotate upwards the $\delta$-images of the elements below $u$. 

![Diagram showing the rotation of $\delta$-images of elements below $u$.]
Dualized construction

The dualized rotation construction obtained, that we denote $S\mathcal{B}(A) \otimes^\Delta S\mathcal{R}(A)$ (where $\Delta = \delta^{-1}$ and $\gamma = \{\nu_b^{-1}\}_{b \in \mathcal{B}(A)}$), is isomorphic to $S(A)$. 

[Diagram of dualized quadruples]
Dualized construction

Our aim is to find the correct definition of abstract dual quadruples, and to understand how to abstractly perform the dualized rotation construction.
Due to our study of introduction, dualization for MTL and GMTL, srDL and dualized quadruples.

**Dualized construction**

Our aim is to find the correct definition of abstract dual quadruples, and to understand how to abstractly perform the dualized rotation construction.

Every prime filter $\alpha$ of an srDL-algebra $A$ contains a unique ultrafilter of its Boolean skeleton.
Our aim is to find the correct definition of abstract dual quadruples, and to understand how to abstractly perform the dualized rotation construction.

Every prime filter $a$ of an srDL-algebra $A$ contains a unique ultrafilter of its Boolean skeleton. Indeed, let $a \in S(A)$, then for each $u \in B(A)$ we have that $u \lor \neg u = 1 \in a$. But $a$ is prime, thus $u \in a$ or $\neg u \in a$. We denote the ultrafilter associated to $a$ by $u_a$. 
**Dualized construction**

Our aim is to find the correct definition of abstract dual quadruples, and to understand how to abstractly perform the dualized rotation construction.

Every prime filter $a$ of an srDL-algebra $A$ contains a unique ultrafilter of its Boolean skeleton. Indeed, let $a \in S(A)$, then for each $u \in B(A)$ we have that $u \lor \neg u = 1 \in a$. But $a$ is prime, thus $u \in a$ or $\neg u \in a$. We denote the ultrafilter associated to $a$ by $u_a$.

For each ultrafilter $u$, set $S_u = \{ a \in S(A) : u_a = u \}$ and call this the site at $u$. 
DUALIZED CONSTRUCTION

Let \( u \) be an ultrafilter of \( \mathcal{B}(A) \), \( \eta \in S(Rad(A)) \).
We say that \( u \) fixes \( \eta \) if there exists \( a \in S_u \) such that \( \eta = a \cap Rad(A) \).
**Dualized construction**

Let $u$ be an ultrafilter of $\mathcal{B}(A)$, $\eta \in S(Rad(A))$. We say that $u$ fixes $\eta$ if there exists $a \in S_u$ such that $\eta = a \cap Rad(A)$.

For every $b \in \mathcal{B}(A)$, let $\mu_b(\eta) = \nu_b^{-1}[\eta] = \{x \in Rad(A) : b \lor x \in \eta\}$. 
Dualized construction

Let \( u \) be an ultrafilter of \( \mathcal{B}(A) \), \( \eta \in S(Rad(A)) \).
We say that \( u \) fixes \( \eta \) if there exists \( a \in S_u \) such that \( \eta = a \cap Rad(A) \).

For every \( b \in \mathcal{B}(A) \), let \( \mu_b(\eta) = \nu_b^{-1}[\eta] = \{ x \in Rad(A) : b \lor x \in \eta \} \).

**Lemma**

*Let \( A \) be an srDL-algebra, \( u \) be an ultrafilter of \( \mathcal{B}(A) \), and \( \eta \in S(Rad(A)) \). The following are equivalent.*

- For every \( b \in \mathcal{B}(A) \) and \( c \in Rad(A) \), if \( b \lor c \in \eta \), then \( b \in u \) or \( c \in \eta \).
- \( u \) fixes \( \eta \).
- \( \mu_b(\eta) = \eta \) for each \( b \notin u \).
EXAMPLE

Let us consider Chang MV-algebra $C$, with domain $C = \{0, c, \ldots, nc, \ldots, 1 - nc, \ldots, 1 - c, 1\}$. $C^+ = \{1, 1 - c, \ldots, 1 - nc, \ldots\}$ is isomorphic to $\mathbb{Z}^-$, and $C$ is isomorphic to the disconnected rotation of $\mathbb{Z}^-$. $C$ generates the variety of perfect MV-algebras.
Example

Let us consider Chang MV-algebra $C$, with domain $C = \{0, c, \ldots, nc, \ldots, 1 - nc, \ldots, 1 - c, 1\}$. $C^+ = \{1, 1 - c, \ldots, 1 - nc, \ldots\}$ is isomorphic to $\mathbb{Z}^-$, and $C$ is isomorphic to the disconnected rotation of $\mathbb{Z}^-$. $C$ generates the variety of perfect MV-algebras.

Consider $C^2 = C \times C$:
The Boolean skeleton of $\mathbb{C}^2$ is the Boolean algebra of 4 elements $\mathcal{B}(\mathbb{C}^2) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

The radical $\mathcal{R}(\mathbb{C}^2)$ is isomorphic to $\mathbb{Z}^- \times \mathbb{Z}^-$, and is the upper square.
We want pairs \((u, r)\) such that for every \(b \notin u\),

\[
\{ y \in \mathcal{R}(C^2) : b \lor y \in r \} = r
\]
We want pairs \((u, r)\) such that for every \(b \notin u\),
\[
\{ y \in \mathcal{R}(C^2) : b \lor y \in r \} = r
\]

Let \(u_1\) be the Boolean ultrafilter generated by \((1, 0)\), \(u_2\) the one generated by \((0, 1)\),
\(Z_1 = \{(1, y) : y \in C^+\}\), and
\(Z_2 = \{(x, 1) : x \in C^+\}\)
We want pairs \((u, r)\) such that for every \(b \notin u\),

\[
\{ y \in R(C^2) : b \lor y \in r \} = r
\]

Let \(u_1\) be the Boolean ultrafilter generated by \((1, 0)\), \(u_2\) the one generated by \((0, 1)\), \(Z_1 = \{(1, y) : y \in C^+ \}\), and \(Z_2 = \{(x, 1) : x \in C^+ \}\).

We get: \((u_2, [r_n])\) and \((u_1, [r_m])\).
We want pairs \((u, \mathfrak{r})\) such that for every \(b \notin u\),
\[\{y \in \mathcal{R}(C^2) : b \lor y \in \mathfrak{r}\} = \mathfrak{r}\]

Let \(u_1\) be the Boolean ultrafilter generated by \((1, 0)\), \(u_2\) the one generated by \((0, 1)\),
\[
Z_1 = \{(1, y) : y \in C^+\}, \text{ and } Z_2 = \{(x, 1) : x \in C^+\}
\]

We get: \((u_2, [r_n])\) and \((u_1, [r_m])\).
Duality for MTL and GMTL

\[
\begin{align*}
(1, 1) & \quad (0, 1) \\
(1, 0) & \quad (0, 1 - nc) \\
(1 - mc, 0) & \quad (0, 1 - nc) \\
(mc, 0) & \quad (0, nc) \\
(0, 0) & \quad (0, nc)
\end{align*}
\]

- \((u_1, \delta[Z_1])\)
- \((u_1, \delta[[r_m]])\)
- \((u_1, \mathcal{R}(C^2))\)
- \((u_1, [r_m])\)
- \((u_1, Z_1)\)
- \((u_2, \delta[Z_2])\)
- \((u_2, \delta[[r_n]])\)
- \((u_2, \mathcal{R}(C^2))\)
- \((u_2, [r_n])\)
- \((u_2, Z_2)\)
Introduction

Duality for MTL and GMTL

srDL and dualized quadruples

\[(1, 1), (0, 0), (1 - mc, 0), (mc, 0), (0, nc), (0, 1 - nc), (0, 1), (1, 0)\]

\[(0, Z_1), (u_1, Z_1), (u_2, Z_1)\]

\[(0, r_m), (u_1, r_m), (u_2, r_m)\]

\[(0, Z_2), (u_1, Z_2), (u_2, Z_2)\]

\[(1 - mc, 0), (mc, 0), (0, nc), (0, 1 - nc)\]

\[(u_1, \delta[Z_1]), (u_2, \delta[Z_2]), (u_1, \delta[Z_1]), (u_2, \delta[Z_2])\]

\[(u_1, \delta[r_m]), (u_2, \delta[r_n]), (u_1, \delta[r_m]), (u_2, \delta[r_n])\]

\[(u_1, R(C^2)), (u_2, R(C^2)), (u_1, R(C^2)), (u_2, R(C^2))\]
Introduction

Duality for MTL and GMTL

srDL and dualized quadruples

\[
\begin{align*}
(1, 1) & \quad (1, 0) \\
(1, 0) & \quad (0, 0) \\
(1 - mc, 0) & \quad (0, 0) \\
(0, 1 - nc) & \quad (0, nc) \\
(0, 1 - nc) & \quad (0, 1) \\
(mc, 0) & \quad (0, nc)
\end{align*}
\]

- \((u_1, \delta[Z_1])\)
- \((u_1, \delta[[r_m]])\)
- \((u_1, \mathcal{R}(\mathcal{C}^2))\)
- \((u_1, [r_m])\)
- \((u_1, Z_1)\)

- \((u_2, \delta[Z_2])\)
- \((u_2, \delta[[r_n]])\)
- \((u_2, \mathcal{R}(\mathcal{C}^2))\)
- \((u_2, [r_n])\)
- \((u_2, Z_2)\)
Dualized Quadruples

Let a *dual quadruple* be a structure \((S, X, \Upsilon, \Delta)\) where

(I) \(S\) is a Stone space;

(II) \(X\) is in GMTL\(^\tau\);

(III) \(\Upsilon = \{\upsilon_U\}_{U \in \mathcal{A}(S)}\) is an indexed family of GMTL\(^\tau\)-morphisms \(\upsilon_U : X \to X\) such that the map \(\vee_e : \mathcal{A}(S) \times \mathcal{A}(X) \to \mathcal{A}(X)\) defined by

\[
\vee_e(U, X) = \upsilon_U^{-1}[X]
\]

is an external join;

(IV) \(\Delta : X \to X\) is a continuous closure operator such that \(R(x, y, z)\) implies \(R(\Delta x, \Delta y, \Delta z)\).
DUALIZED QUADRUPLES

Given a dual quadruple $(S, X, \gamma, \Delta)$, let $S \otimes_{\gamma} X$ be the structure obtained by the dual rotation construction, with suitably defined topology and multiplication.

**Theorem**

Let $(S, X, \gamma, \Delta)$ be a dual quadruple. Then $S \otimes_{\gamma} X$ is the extended Priestley dual of some srDL-algebra.

**Theorem**

Let $Y$ be the extended Priestley dual of an srDL-algebra. Then there exists a dual quadruple $(S, X, \gamma, \Delta)$ such that $Y \cong S \otimes_{\gamma} X$. 
Thank you!