Section I: Some fact about Boolean rings

1. A ring $R$ is called Boolean if $x^2 = x$ for all $x \in R$. Prove that if $R$ is a Boolean ring, then $R$ is commutative. Need a Boolean ring have a 1? Can a Boolean ring be an integral domain?

2. Prove that in a Boolean ring every prime ideal is a maximal ideal.

3. Let $R$ be a Boolean ring and $I \subseteq R$ an ideal. Define a function $f : R \to \mathbb{Z}_2$ by $f(x) = 0$ iff $x \in I$ and $f(x) = 1$ otherwise. Prove that $f$ is a ring homomorphism iff $I$ is a prime ideal.

Section II: Zorn’s lemma and its applications

Let $(P, \leq)$ be a partially-ordered set. A subset $C \subseteq P$ is called a chain if for each $x, y \in P$ we have that $x \leq y$ or $y \leq x$. Zorn’s lemma states that if each chain contained in $P$ has an upper bound in $P$, then $P$ has a maximal element (see Appendix I of Dummit and Foote for general information, and the proof of Proposition 11 on p. 254 for a typical application). This has a huge number of applications in ring theory, a few of which we sketch below.

4. Let $A = (a_1, a_2, ..., a_n)$ be a nonzero finitely generated ideal of the ring $R$. Prove that there is an ideal $B$ which is maximal with respect to the property that it does not contain $A$.

5. Let $R$ be a commutative ring. Prove that the set of prime ideals in $R$ has a minimal element with respect to inclusion (possibly the zero ideal).

6. Let $S$ be a multiplicatively closed subset of a commutative ring $R$ such that $0 \notin S$. Prove that there is an ideal $I$ that is maximal with respect to the condition that $I \cap S = \emptyset$. 