1. A topological space \((X, \tau)\) is called Lindelöf if every open cover of \(X\) has a countable subcover.  
(a) Prove that a metric space is Lindelöf if and only if it is separable.  
(b) Show that an arbitrary subspace of a Lindelöf space need not be Lindelöf, but that a closed subspace of a Lindelöf space is Lindelöf.

2. Prove the Alexander subbase theorem: Let \((X, \tau)\) is a topological space with subbase \(\mathcal{B}\). If every cover of \(X\) by elements of \(\mathcal{B}\) has a finite subcover, then \((X, \tau)\) is compact.

3. A topological space is called locally compact if every point of \(X\) has a compact neighborhood.  
(a) Show that \(\mathbb{R}\) is locally compact, but that \(\mathbb{Q}\) is not.  
(b) Let \((X, \tau)\) be a topological space, and define \(X^* = X \cup \{\infty\}\) and  
\[
\tau^* = \tau \cup \{(X \setminus K) \cup \{\infty\} : K \subseteq X \text{ is closed and compact}\}.
\]
Show that \(X^*\) is compact, that \(X\) is an open subspace of \(X^*\), and that \(X^*\) is Hausdorff if and only if \(X\) is Hausdorff and locally compact.  
(c) Show that a topological space is homeomorphic to an open subspace of a compact Hausdorff space if and only if it is locally compact and Hausdorff.